9. Asymptotics

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Gov 2002 (Harvard)

Where are we? Where are we going?

- Last time: introducing estimators, looking at finite-sample properties.
- Now: can we say more as sample size grows?

1/ Asymptotics

Current knowledge

- For i.i.d. r.v.s, $X_1, \ldots, X_n$, with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2$ we know that:
    - $\overline{X}_n$ is **unbiased**, $\mathbb{E}[\overline{X}_n] = \mathbb{E}[X_i] = \mu$
    - Sampling variance is $\mathbb{V}[\overline{X}_n] = \frac{\sigma^2}{n}$ where $\sigma^2 = \mathbb{V}[X_i]$
    - None of these rely on a **specific distribution** for X_i!
- Assuming $X_i \sim \mathcal{N}(\mu, \sigma^2)$, we know the exact distribution of $\overline{X}_n$.
    - What if the data isn't normal? What is the sampling distribution of $\overline{X}_n$?
- **Asymptotics**: approximate the sampling distribution of $\overline{X}_n$ as n gets big.

Sequence of sample means

- What can we say about the sample mean n gets large?
- Need to think about sequences of sample means with increasing n:

$$
\begin{aligned}
\overline{X}_1 &= X_1 \\
\overline{X}_2 &= (1/2) \cdot (X_1 + X_2) \\
\overline{X}_3 &= (1/3) \cdot (X_1 + X_2 + X_3) \\
\overline{X}_4 &= (1/4) \cdot (X_1 + X_2 + X_3 + X_4) \\
\overline{X}_5 &= (1/5) \cdot (X_1 + X_2 + X_3 + X_4 + X_5) \\
&\vdots \\
\overline{X}_n &= (1/n) \cdot (X_1 + X_2 + X_3 + X_4 + X_5 + \cdots + X_n)
\end{aligned}
$$

- Note: this is a sequence of random variables!

Asymptotics and Limits

- Asymptotic analysis is about making **approximations** to finite sample properties.
- Useful to know some properties of deterministic sequences:

Definition

A sequence $\{a_n : n = 1, 2, \ldots\}$ has the **limit** a written $a_n \to a$ as $n \to \infty$ if for all $\delta > 0$ there is some $n_\delta < \infty$ such that for all $n \geq n_\delta$, $|a_n - a| \leq \delta$.

- a_n gets closer and closer to a as n gets larger (a_n **converges** to a)
- $\{a_n : n = 1, 2, \ldots\}$ is **bounded** if there is $b < \infty$ such that $|a_n| < b$ for all n.

Convergence in Probability

Definition

A sequence of random variables, $\{Z_n : n = 1, 2, ...\}$, is said to **converge in probability** to a value b if for every $\varepsilon > 0$,

$$\mathbb{P}(|Z_n - b| > \varepsilon) \to 0,$$

as $n \to \infty$. We write this $Z_n \xrightarrow{p} b$.

- Basically: probability that Z_n lies outside any (teeny, tiny) interval around b approaches 0 as $n \to \infty$
- Economists writes $\text{plim}(Z_n) = b$ if $Z_n \xrightarrow{p} b$.
- An estimator is **consistent** if $\hat{\theta}_n \xrightarrow{p} \theta$.
  - Distribution of $\hat{\theta}_n$ collapses on θ as $n \to \infty$.
  - Inconsistent estimator are bad bad bad: more data gives worse answers!

Chebyshev Inequality

- How can we show convergence in probability? Can verify if we know specific distribution of $\hat{\theta}$.
- But can we say anything for arbitrary distributions?

Chebyshev Inequality

Suppose that X is r.v. for which $\mathbb{V}[X] < \infty$. Then, for every real number $\delta > 0$,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq \delta) \leq \frac{\mathbb{V}[X]}{\delta^2}.$$

- Variance places limits on how far an observation can be from its mean.

Proof of Chebyshev

- Let $Z = X - \mathbb{E}[X]$ with density $f_Z(x)$. Probability is just integral over the region:

$$\mathbb{P}\left(|Z| \geq \delta\right) = \int_{|x|\geq\delta} f_Z(x)dx$$

- Note that where $|x| \geq \delta$, we have $1 \leq x^2/\delta^2$, so

$$\mathbb{P}\left(|Z| \geq \delta\right) \leq \int_{|x|\geq\delta} \frac{x^2}{\delta^2} f_Z(x)dx \leq \int_{-\infty}^{\infty} \frac{x^2}{\delta^2} f_Z(x)dx = \frac{\mathbb{E}[Z^2]}{\delta^2} = \frac{\mathbb{V}[X]}{\delta^2}$$

Law of large numbers

Weak Law of Large Numbers

Let $X_1, \ldots, X_n$ be a an i.i.d. draws from a distribution with mean $\mathbb{E}[|X_i|] < \infty$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Then, $\overline{X}_n \overset{p}{\to} \mathbb{E}[X_i]$.

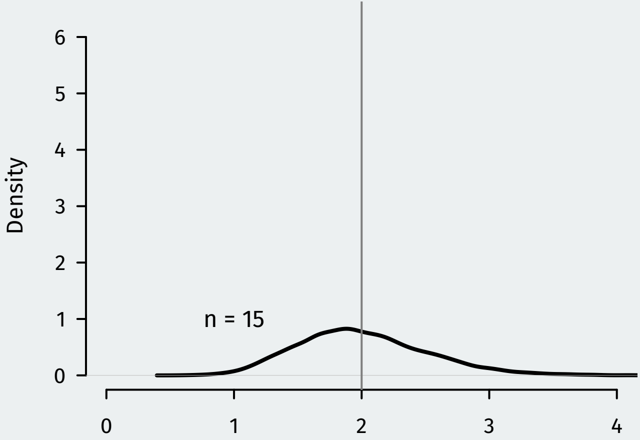
- Note: we don't assume finite variance, only finite expectation.
    - Proof with finite variance is an easy application of Chebyshev.
- Intuition: The probability of $\overline{X}_n$ being "far away" from μ goes to 0 as n gets big.
- Implies general consistency of **plug-in estimators**
    - If $\mathbb{E}[|g(X_i)|] < \infty$, then $\frac{1}{n}\sum_{i=1}^n g(X_i) \overset{p}{\to} \mathbb{E}[g(X_i)]$

LLN by simulation in R

- Draw different sample sizes from Exponential distribution with rate 0.5
- $\rightsquigarrow \mathbb{E}[X_i] = 2$

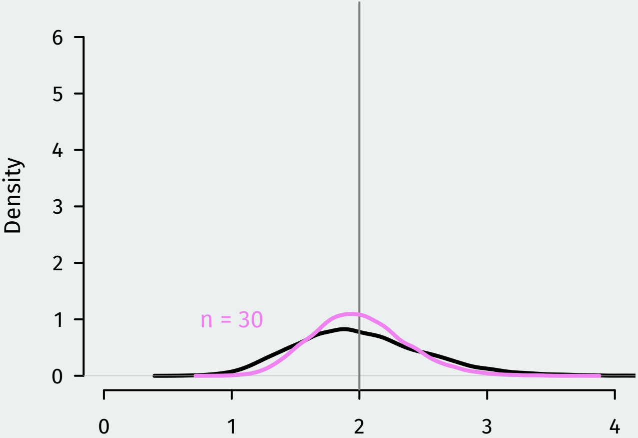
```
nsims <- 10000
holder <- matrix(NA, nrow = nsims, ncol = 6)
for (i in 1:nsims) {
  s5 <- rexp(n = 5, rate = 0.5)
  s15 <- rexp(n = 15, rate = 0.5)
  s30 <- rexp(n = 30, rate = 0.5)
  s100 <- rexp(n = 100, rate = 0.5)
  s1000 <- rexp(n = 1000, rate = 0.5)
  s10000 <- rexp(n = 10000, rate = 0.5)

  holder[i,1] <- mean(s5)
  holder[i,2] <- mean(s15)
  holder[i,3] <- mean(s30)
  holder[i,4] <- mean(s100)
  holder[i,5] <- mean(s1000)
  holder[i,6] <- mean(s10000)
}
```

LLN in action



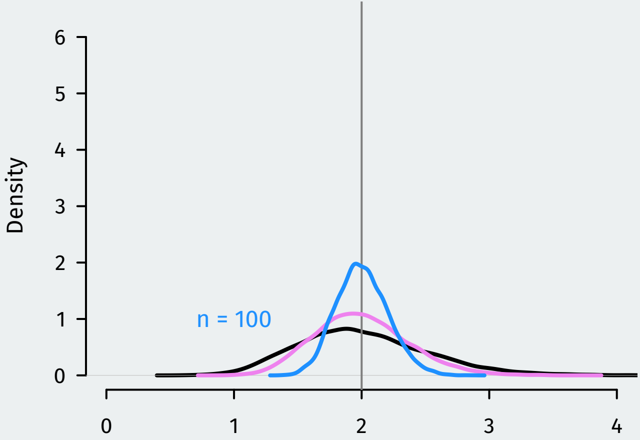
- Distribution of $\overline{X}_{15}$

LLN in action



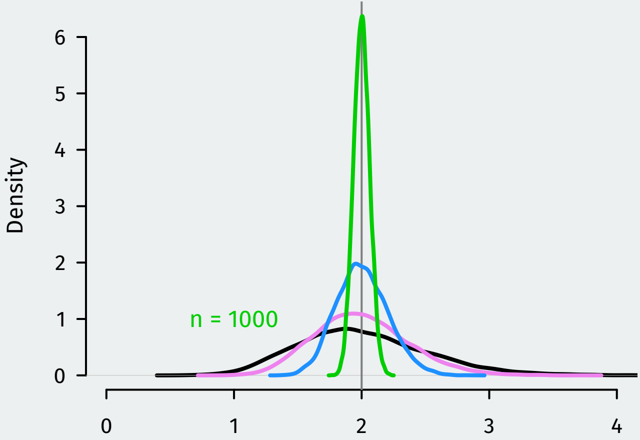
- Distribution of $\overline{X}_{30}$

LLN in action



- Distribution of $\overline{X}_{100}$

LLN in action



- Distribution of $\overline{X}_{1000}$

Properties of convergence in probability

1. **Continuous mapping theorem**: if $X_n \overset{p}{\to} c$, then $g(X_n) \overset{p}{\to} g(c)$ for any continuous function g.
2. if $X_n \overset{p}{\to} a$ and $Z_n \overset{p}{\to} b$, then
   - $X_n + Z_n \overset{p}{\to} a + b$
   - $X_n Z_n \overset{p}{\to} ab$
   - $X_n / Z_n \overset{p}{\to} a/b$ if $b > 0$

- Thus, by LLN and CMT:
  - $(\overline{X}_n)^2 \overset{p}{\to} \mu^2$
  - $\log(\overline{X}_n) \overset{p}{\to} \log(\mu)$

Unbiased versus consistent

- By Chebyshev, unbiased estimators are consistent if $\mathbb{V}[\hat{\theta}_n] \to 0$.
- **Unbiased, not consistent**: "first observation" estimator, $\hat{\theta}_n^f = X_1$.
  - Unbiased because $\mathbb{E}[\hat{\theta}_n^f] = \mathbb{E}[X_1] = \mu$
  - Not consistent: $\hat{\theta}_n^f$ is constant in n so its distribution never collapses.
  - Said differently: the variance of $\hat{\theta}_n^f$ never shrinks.
- **Consistent, but biased**: sample mean with n replaced by $n - 1$:

$$\frac{1}{n-1}\sum_{i=1}^{n} X_i = \frac{n}{n-1}\overline{X}_n \overset{p}{\to} 1 \times \mu$$

  - Consistent because $n/(n-1) \to 1$ as $n \to \infty$.

Multivariate LLN

- Let $\mathbf{X}_i = (X_{i1}, \dots, X_{ik})$ be a random vectors of length k.
- Random (iid) sample of n of these k vectors, $\mathbf{X}_1, \dots, \mathbf{X}_n$.
- Vector sample mean:

$$\overline{\mathbf{X}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i = \begin{pmatrix} \overline{X}_{n,1} \\ \overline{X}_{n,2} \\ \vdots \\ \overline{X}_{n,k} \end{pmatrix}$$

- **Vector WLLN**: if $\mathbb{E}[\|\mathbf{X}\|] < \infty$, then as $n \to \infty$, $\overline{\mathbf{X}}_n \overset{p}{\to} \mathbb{E}[\mathbf{X}]$.
    - Converge in probability of a vector is just convergence of each element.
    - $\mathbb{E}[\|\mathbf{X}\|] < \infty$ is equivalent to $\mathbb{E}[|X_{ij}|] < \infty$ for each $j = 1, \dots, k$

2/ Central Limit Theorem

Current knowledge

- For i.i.d. r.v.s, $X_1, \ldots, X_n$, with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2$ we know that:
  - $\mathbb{E}[\overline{X}_n] = \mu$ and $\mathbb{V}[\overline{X}_n] = \frac{\sigma^2}{n}$
  - $\overline{X}_n$ converges to μ as n gets big
  - Chebyshev provides some bounds on probabilities.
  - Still no distributional assumptions about X_i!
- Can we say more?
  - Can we approximate $\Pr(a < \overline{X}_n < b)$?
  - What family of distributions (Binomial, Uniform, Gamma, etc)?
- Again, need to analyze when n is large.

Convergence in Distribution

Definition

Let $Z_1, Z_2, \ldots$, be a sequence of r.v.s, and for $n = 1, 2, \ldots$ let $F_n(u)$ be the c.d.f. of Z_n. Then it is said that $Z_1, Z_2, \ldots$ **converges in distribution** to r.v. W with c.d.f. $F_W(u)$ if

$$\lim_{n\to\infty} F_n(u) = F_W(u),$$

which we write as $Z_n \overset{d}{\to} W$.

- Basically: when n is big, the distribution of Z_n is very similar to the distribution of W
    - Also known as the **asymptotic distribution** or **large-sample distribution**
- We use c.d.f.s here to avoid messy details with discrete vs continuous.
- If $X_n \overset{p}{\to} X$, then $X_n \overset{d}{\to} X$

Central Limit Theorem

Central Limit Theorem

Let $X_1, \ldots, X_n$ be i.i.d. r.v.s from a distribution with mean $\mu = \mathbb{E}[X_i]$ and variance $\sigma^2 = \mathbb{V}[X_i]$. Then if $\mathbb{E}[X_i^2] < \infty$, we have

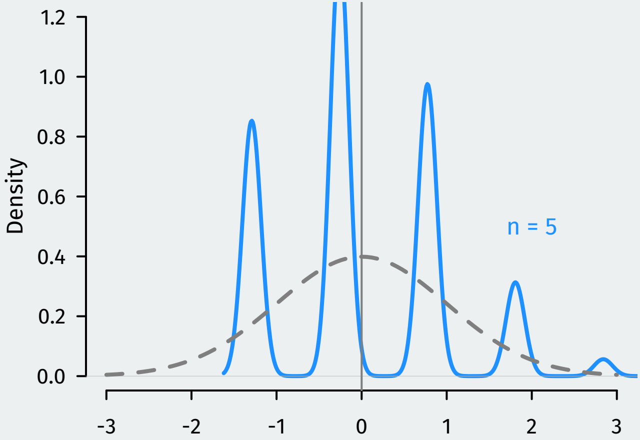
$$\sqrt{n}\left(\overline{X}_n - \mu\right) \overset{d}{\to} \mathcal{N}(0, \sigma^2).$$

- Subtle point: why center and scale by $\sqrt{n}$?
  - The LLN implied that $\overline{X}_n \overset{p}{\to} \mu$ so $\overline{X}_n \overset{d}{\to} \mu$, which isn't very helpful!
  - $\sqrt{n}\left(\overline{X}_n - \mu\right)$ is more "stable" since its variance doesn't depend on n
- But we can use the result to get an approximation: $\overline{X}_n \overset{a}{\sim} N(\mu, \sigma^2/n)$,
  - $\overset{a}{\sim}$ is "approximately distributed as".
- No assumptions about the distribution of X_i except finite variance.
- $\rightsquigarrow$ approximations to probability statements about $\overline{X}_n$ when n is big!

CLT by simulation in R

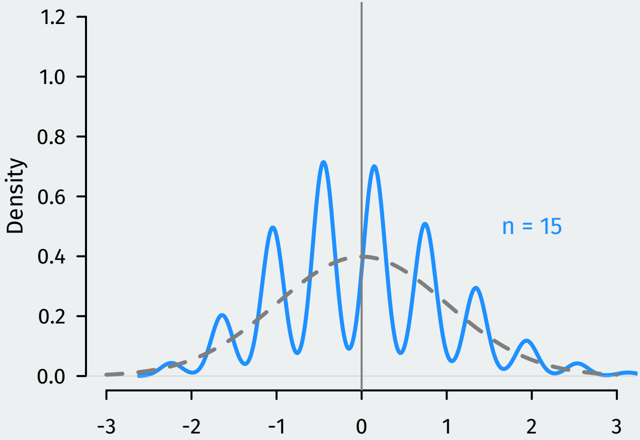
```r
set.seed(02138)
nsims <- 10000
holder2 <- matrix(NA, nrow = nsims, ncol = 6)
for (i in 1:nsims) {
  s5 <- rbinom(n = 5, size = 1, prob = 0.25)
  s15 <- rbinom(n = 15, size = 1, prob = 0.25)
  s30 <- rbinom(n = 30, size = 1, prob = 0.25)
  s100 <- rbinom(n = 100, size = 1, prob = 0.25)
  s1000 <- rbinom(n = 1000, size = 1, prob = 0.25)
  s10000 <- rbinom(n = 10000, size = 1, prob = 0.25)

  holder2[i,1] <- mean(s5)
  holder2[i,2] <- mean(s15)
  holder2[i,3] <- mean(s30)
  holder2[i,4] <- mean(s100)
  holder2[i,5] <- mean(s1000)
  holder2[i,6] <- mean(s10000)
}
```

CLT in action



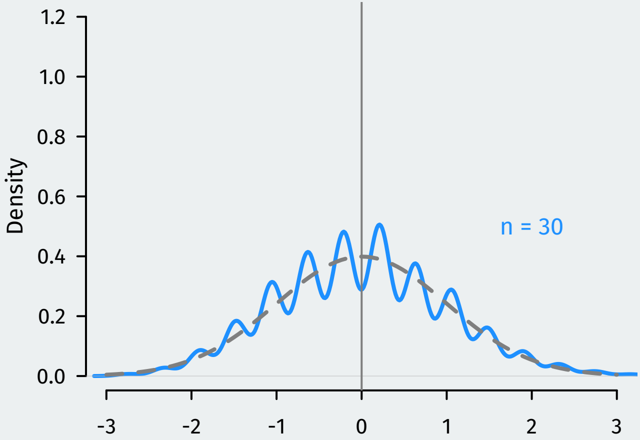
- Distribution of $\frac{\overline{X}_5 - \mu}{\sigma/\sqrt{5}}$

CLT in action



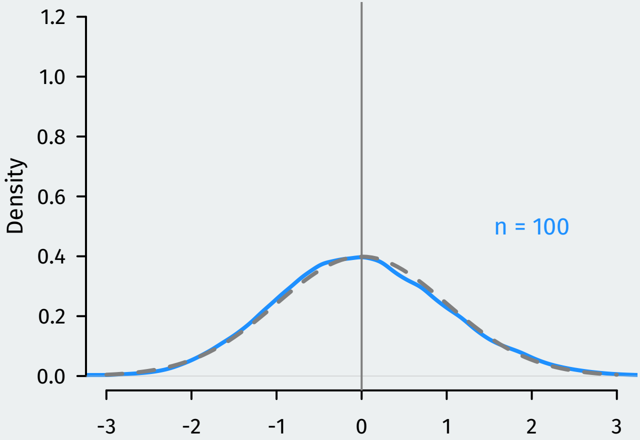
- Distribution of $\frac{\overline{X}_{15}-\mu}{\sigma/\sqrt{15}}$

CLT in action



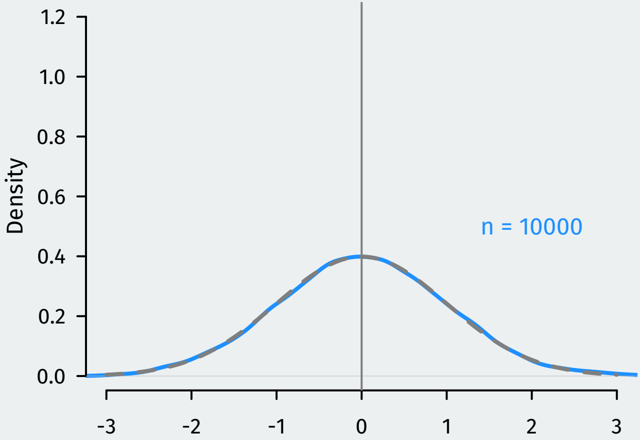
- Distribution of $\frac{\overline{X}_{30} - \mu}{\sigma/\sqrt{30}}$

CLT in action



- Distribution of $\frac{\overline{X}_{100} - \mu}{\sigma / \sqrt{100}}$

CLT in action



- Distribution of $\frac{\overline{X}_{10000}-\mu}{\sigma/\sqrt{10000}}$

Transformations

- Continuous mapping theorem: for continuous g, we have

$$Z_n \xrightarrow{d} Z \quad \Longrightarrow \quad g(Z_n) \xrightarrow{d} g(Z).$$

- Let $X_1, X_2, \ldots$ converge in distribution to some r.v. X
- Let $Y_1, Y_2, \ldots$ converge in probability to some number, c
- Slutsky's Theorem gives the following result:
  1. $X_n Y_n$ converges in distribution to cX
  2. $X_n + Y_n$ converges in distribution to $X + c$
  3. X_n / Y_n converges in distribution to X/c if $c \neq 0$
- Extremely useful when trying to figure out what the large-sample distribution of an estimator is.

Asymptotic normality

- An estimator $\hat{\theta}_n$ for θ is **asymptotically normal** when

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \overset{d}{\to} \mathcal{N}(0, V_\theta)$$

  - Sample mean: $\sqrt{n}(\overline{X}_n - \mu) \overset{d}{\to} \mathcal{N}(0, \sigma^2)$
  - Usually follows from some version of the CLT

- V_θ is the variance of this centered/scaled version of the estimator.
  - The approximate variance of the estimator itself will be $\mathbb{V}[\hat{\theta}_n] \overset{a}{=} V_\theta/n$
  - The approximate **standard error** will be $\text{se}[\hat{\theta}_n] = \sqrt{V/n}$

- Allows us to approximate the probability of $\hat{\theta}_n$ being far away from θ in large samples.
  - **Warning:** you do not know if you sample is big enough for this to be a good approximation.

Variance estimation with plug-in estimators

- Setting: $X_1, \ldots, X_n$ i.i.d. with quantity of interest $\theta = \mathbb{E}[g(X_i)]$
  - Let $V_\theta = \mathbb{V}[g(X_i)] = \mathbb{E}[(g(X_i) - \theta)^2]$.
- Analogy/plug-in estimator: $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^n g(X_i)$
- By the CLT, if $\mathbb{E}[g(X_i)^2] < \infty$ then

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, V_\theta)$$

- But we don't know V_θ?! Estimate it!

$$\widehat{V}_\theta = \frac{1}{n}\sum_{i=1}^n \left(g(X_i) - \hat{\theta}_n\right)^2$$

- We can show that $\widehat{V}_\theta \xrightarrow{p} V_\theta$ and so by Slutsky:

$$\frac{\sqrt{n}\left(\hat{\theta}_n - \theta\right)}{\sqrt{\widehat{V}_\theta}} \xrightarrow{d} \frac{\mathcal{N}(0, V_\theta)}{\sqrt{V_\theta}} \sim \mathcal{N}(0, 1)$$

Multivariate CLT

- Convergence in distribution is the same vector $\mathbf{Z}_n$: convergence of c.d.f.s
- Allow us to generalize the CLT to random vectors:

Multivariate Central Limit Theorem

If $\mathbf{X}_i \in \mathbb{R}^k$ are i.i.d. and $\mathbb{E}\|\mathbf{X}_i\|^2 < \infty$, then as $n \to \infty$,

$$\sqrt{n}\left(\overline{\mathbf{X}}_n - \boldsymbol{\mu}\right) \overset{d}{\to} \mathcal{N}(0, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\mu} = \mathbb{E}[\mathbf{X}_i]$ and $\boldsymbol{\Sigma} = \mathbb{V}[\mathbf{X}_i] = \mathbb{E}\left[(\mathbf{X}_i - \boldsymbol{\mu})(\mathbf{X}_i - \boldsymbol{\mu})'\right]$.

- $\mathbb{E}\|\mathbf{X}_i\|^2 < \infty$ is equivalent to $\mathbb{E}[X_{i,j}^2] < \infty$ for all $j = 1, \ldots, k$.
  - Basically: multivariate CLT holds if each r.v. in the vector has finite variance.
- Very common for when we're estimating multiple parameters $\boldsymbol{\theta}$ with $\widehat{\boldsymbol{\theta}}_n$

3/ Confidence intervals

Interval estimation - what and why?

- $\hat{\theta}_n$ is our best guess about θ
- But $\mathbb{P}(\hat{\theta}_n = \theta) = 0$!
- Alternative: produce a range of plausible values instead of one number.
    - Hopefully will increase the chance that we've captured the truth.
- We can use the distribution of estimators (CLT!!) to derive these intervals.

What is a confidence interval?

Definition

A $1 - \alpha$ **confidence interval** for a population parameter θ is a pair of statistics $L = L(X_1, \dots, X_n)$ and $U = U(X_1, \dots, X_n)$ such that $L < U$ and such that

$$\mathbb{P}(L \leq \theta \leq U) = 1 - \alpha, \quad \forall \theta$$

- Random interval (L, U) will contain the truth $1 - \alpha$ of the time.
  - $\mathbb{P}(L \leq \theta \leq U)$ is the **coverage probability** of the CI
- Extremely useful way to represent our uncertainty about our estimate.
  - Shows a range of plausible values given the data.
- A sequence of CIs, $[L_n, U_n]$ are **asymptotically valid** if the coverage probability converges to correct level:

$$\lim_{n \to \infty} \mathbb{P}(L_n \leq \theta \leq U_n) = 1 - \alpha$$

Asymptotic confidence intervals

- A sequence of CIs, $[L_n, U_n]$ are **asymptotically valid** if the coverage probability converges to correct level:

$$\lim_{n \to \infty} \mathbb{P}(L_n \leq \theta \leq U_n) = 1 - \alpha$$

- We can derive such CIs when our estimators are asymptotically normal:

$$\frac{\hat{\theta}_n - \theta}{\widehat{\text{se}}(\hat{\theta}_n)} \overset{d}{\to} \mathcal{N}(0, 1)$$

- Then as $n \to \infty$

$$\mathbb{P}\left(-1.96 \leq \frac{\hat{\theta}_n - \theta}{\widehat{\text{se}}(\hat{\theta})} \leq 1.96\right) \to 0.95$$

Deriving the 95% CI

$$\mathbb{P}\left(-1.96 \leq \frac{\hat{\theta}_n - \theta}{\widehat{\text{se}}(\hat{\theta}_n)} \leq 1.96\right) \to 0.95$$

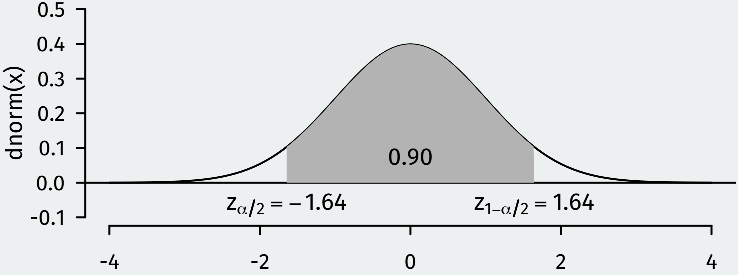
$$\mathbb{P}\left(-1.96 \cdot \widehat{\text{se}}(\hat{\theta}_n) \leq \hat{\theta}_n - \theta \leq 1.96 \cdot \widehat{\text{se}}(\hat{\theta}_n)\right) \to 0.95$$

$$\mathbb{P}\left(-\hat{\theta}_n - 1.96 \cdot \widehat{\text{se}}(\hat{\theta}_n) \leq -\theta \leq -\hat{\theta}_n + 1.96 \cdot \widehat{\text{se}}(\hat{\theta}_n)\right) \to 0.95$$

$$\mathbb{P}\left(\hat{\theta}_n - 1.96 \cdot \widehat{\text{se}}(\hat{\theta}_n) \leq \theta \leq \hat{\theta}_n + 1.96 \cdot \widehat{\text{se}}(\hat{\theta}_n)\right) \to 0.95$$

- Lower bound: $\hat{\theta}_n - 1.96 \cdot \text{se}(\hat{\theta}_n)$
- Upper bound: $\hat{\theta}_n + 1.96 \cdot \text{se}(\hat{\theta}_n)$

Finding the critical values



$$\mathbb{P}\left(-z_{1-\alpha/2} \leq \frac{\hat{\theta}_n - \theta}{\widehat{\text{se}}(\hat{\theta}_n)} \leq z_{1-\alpha/2}\right) \to 1-\alpha \quad \Longrightarrow \quad (1-\alpha)\text{ CI: } \hat{\theta}_n \pm z_{1-\alpha/2} \cdot \widehat{\text{se}}(\hat{\theta}_n)$$

- How do we figure out what $z_{1-\alpha/2}$ will be?
- Intuitively, we want the z values that puts $\alpha/2$ in each of the tails.
  - Because normal is symmetric, we have $z_{\alpha/2} = -z_{1-\alpha/2}$
  - Use the quantile function: $z_{1-\alpha/2} = \Phi^{-1}(1-\alpha/2)$ (`qnorm` in R)

CI for social pressure effect

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

	Experimental Group				
	Control	Civic Duty	Hawthorne	Self	Neighbors
Percentage Voting	29.7%	31.5%	32.2%	34.5%	37.8%
N of Individuals	191,243	38,218	38,204	38,218	38,201

```
neigh_var <- var(social$voted[social$treatment == "Neighbors"])
neigh_n <- 38201
civic_var <- var(social$voted[social$treatment == "Civic Duty"])
civic_n <- 38218

se_diff <- sqrt(neigh_var/neigh_n + civic_var/civic_n)

## c(lower, upper)
c((0.378 - 0.315) - 1.96 * se_diff, (0.378 - 0.315) + 1.96 * se_diff)
```

```
## [1] 0.0563 0.0697
```

Interpreting the confidence interval

- **Caution:** a common **incorrect** interpretation of a confidence interval:
  - "I calculated a 95% confidence interval of [0.05,0.13], which means that there is a 95% chance that the true difference in means in is that interval."
  - This is WRONG.
- The true value of the population mean, μ, is **fixed**.
  - It is either in the interval or it isn't—there's no room for probability at all.
- The randomness is in the interval: $\overline{X}_n \pm 1.96 S_n/\sqrt{n}$.
- Correct interpretation: **across 95% of random samples, the constructed confidence interval will contain the true value.**

Confidence interval simulation

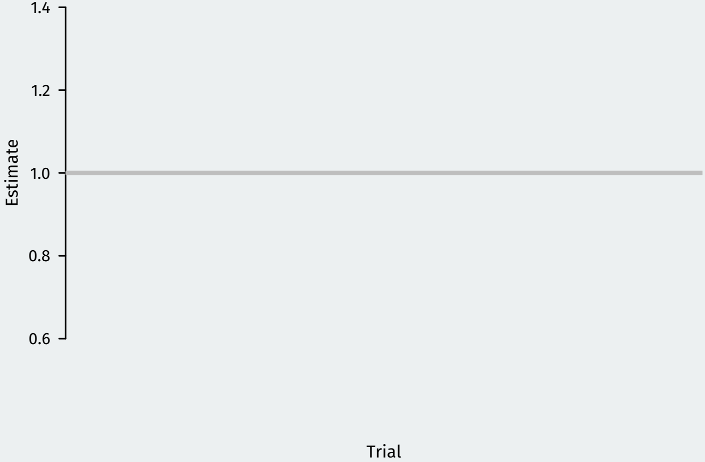
- Draw samples of size 500 (pretty big) from $\mathcal{N}(1, 10)$
- Calculate confidence intervals for the sample mean:

$$\overline{X}_n \pm 1.96 \times \widehat{\text{se}}[\overline{X}_n] \rightsquigarrow \overline{X}_n \pm 1.96 \times S_n/\sqrt{n}$$

```
sims<- 10000
cover <- rep(0, times = sims)
low.bound <- up.bound <- rep(NA, times = sims)
for(i in 1:sims){
    draws <- rnorm(500, mean = 1, sd = sqrt(10))
    low.bound[i] <- mean(draws) - sd(draws) / sqrt(500) * 1.96
    up.bound[i] <- mean(draws) + sd(draws) / sqrt(500) * 1.96
    if (low.bound[i] < 1 & up.bound[i] > 1) {
        cover[i] <- 1
    }
}
mean(cover)
```

```
## [1] 0.95
```

Plotting the CIs



Plotting the CIs

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Plotting the CIs

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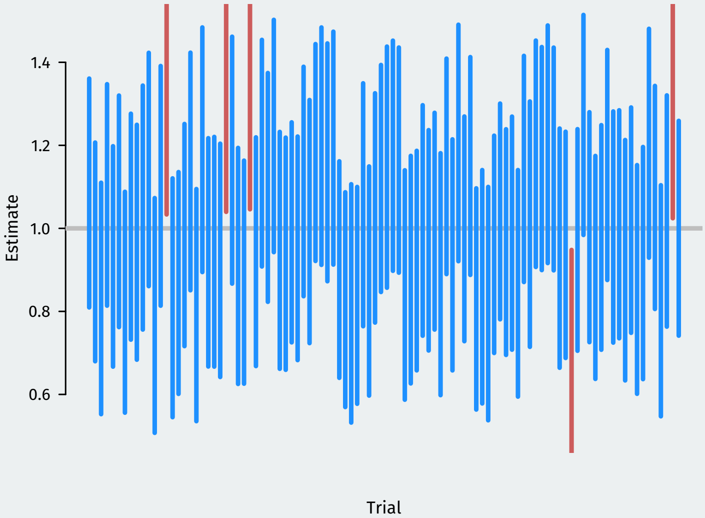

Question

- **Question** What happens to the size of the confidence interval when we increase our confidence, from say 95% to 99%? Do confidence intervals get wider or shorter?
- **Answer** Wider!
- Decreases $\alpha \rightsquigarrow$ increases $1 - \alpha/2 \rightsquigarrow$ increases $z_{\alpha/2}$

4/ Delta method

Delta method

Delta method

If $\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, V_\theta)$ and $h(u)$ is continuously differentiable in a neighborhood around θ, then as $n \to \infty$,

$$\sqrt{n}\left(h(\hat{\theta}_n) - h(\theta)\right) \xrightarrow{d} \mathcal{N}(0, (h'(\theta))^2 V_\theta).$$

- Why $h()$ continuously differentiable?
  - Near θ we can approximate $h()$ with a line where h' is the slope.
  - So $h(\hat{\theta}_n) - h(\theta) \approx h'(\theta)\left(\hat{\theta}_n - \theta\right)$
- Examples:
  - $\sqrt{n}(\overline{X}_n^2 - \mu^2) \xrightarrow{d} \mathcal{N}(0, (2\mu)^2\sigma^2)$
  - $\sqrt{n}(\log(\overline{X}_n) - \log(\mu)) \xrightarrow{d} \mathcal{N}(0, \sigma^2/\mu^2)$

Multivariate Delta Method

- What if we want to know the asymptotic distribution of a function of $\hat{\boldsymbol{\theta}}_n$?
- Let $\mathbf{h}(\boldsymbol{\theta})$ map from $\mathbb{R}^k \to \mathbb{R}^m$ and be continuously differentiable.
  - Ex: $\mathbf{h}(\theta_1, \theta_2, \theta_3) = (\theta_2/\theta_1, \theta_3/\theta_1)$, from $\mathbb{R}^3 \to \mathbb{R}^2$
  - Like univariate case, we need the derivatives arranged in $m \times k$ Jacobian matrix:

$$\mathbf{H}(\boldsymbol{\theta}) = \boldsymbol{\nabla}_{\boldsymbol{\theta}}\mathbf{h}(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial h_1}{\partial \theta_1} & \frac{\partial h_1}{\partial \theta_2} & \cdots & \frac{\partial h_1}{\partial \theta_k} \\ \frac{\partial h_2}{\partial \theta_1} & \frac{\partial h_2}{\partial \theta_2} & \cdots & \frac{\partial h_2}{\partial \theta_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial \theta_1} & \frac{\partial h_m}{\partial \theta_2} & \cdots & \frac{\partial h_m}{\partial \theta_k} \end{pmatrix}$$

- Multivariate delta method: if $\sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right) \overset{d}{\to} \mathcal{N}(0, \boldsymbol{\Sigma})$, then

$$\sqrt{n}\left(\mathbf{h}(\hat{\boldsymbol{\theta}}_n) - \mathbf{h}(\boldsymbol{\theta})\right) \overset{d}{\to} \mathcal{N}(0, \mathbf{H}(\boldsymbol{\theta})\boldsymbol{\Sigma}\mathbf{H}(\boldsymbol{\theta})')$$

Stochastic order notation

- When working with asymptotics, it's often useful to have some shorthand.
- Order notation for deterministic sequences:
    - If $a_n \to 0$, then we write $a_n = o(1)$ ("little-oh-one")
    - If $n^{-\lambda} a_n \to 0$, we write $a_n = o(n^\lambda)$
    - If a_n is bounded, we write $a_n = O(1)$ ("big-oh-one")
    - If $n^{-\lambda} a_n$ is bounded, we write $a_n = O(n^\lambda)$
- Stochastic order notation for random sequence, Z_n
    - If $Z_n \overset{p}{\to} 0$, we write $Z_n = o_p(1)$ ("little-oh-p-one").
    - For any consistent estimator, we have $\hat{\theta}_n = \theta + o_p(1)$
    - If $a_n^{-1} Z_n \overset{p}{\to} 0$, we write $Z_n = o_p(a_n)$

Bounded in probability

Definition

A random sequence Z_n is **bounded in probability**, written $Z_n = O_p(1)$ ("big-oh-p-one") for all $\delta > 0$ there exists a M_δ and n_δ, such that for $n \geq n_\delta$,

$$\mathbb{P}(|Z_n| > M_\delta) < \delta$$

- $Z_n = o_p(1)$ implies $Z_n = O_p(1)$ but not the reverse.
- If Z_n converges in distribution, it is $O_p(1)$, so if the CLT applies we have:

$$\sqrt{n}(\hat{\theta}_n - \theta) = O_p(1)$$

  - If $a_n^{-1} Z_n = O_p(1)$, we write $Z_n = O_p(a_n)$, so we have: $\hat{\theta}_n = \theta + O_p(n^{-1/2})$.