1: Basic Probability

Spring 2023

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Gov 2002 (Harvard)

What is a reasonably safe gathering size in the age of COVID?

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  - The heart of **statistical inference**

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- Learning mathematical probability avoids these mistakes!

Learning about populations

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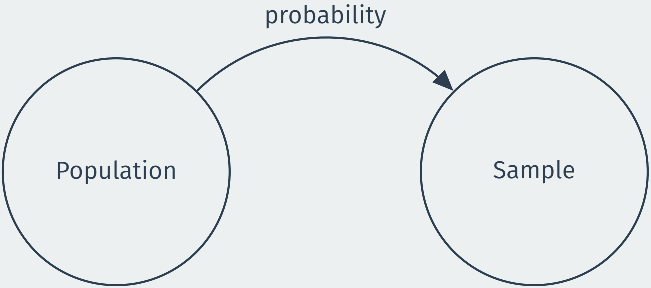

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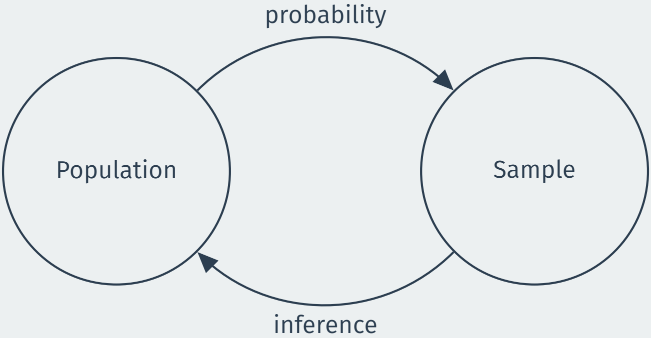

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- **Inference**: learning about the population from a set of data.

Roadmap

1. Naive Definition of Probability

2. Non-naive Definition of Probability

1/ Naive Definition of Probability

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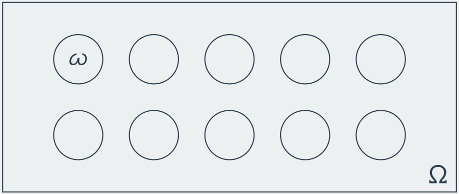

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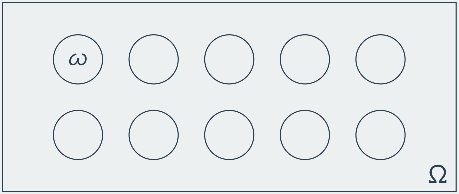

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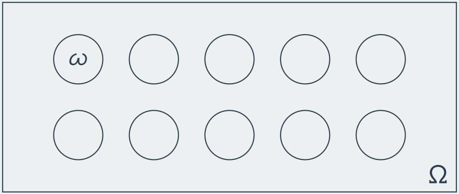

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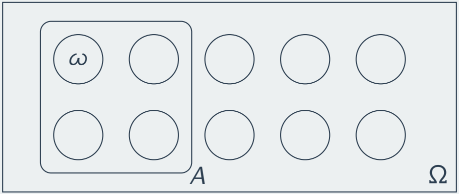

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- $\omega \in \Omega$ is one particular **outcome**.
- A subset of Ω is an **event** and we write this as $A \subset \Omega$.

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$$\mathbb{P}(4\clubsuit) = \mathbb{P}(4\heartsuit) = 1/52$$

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- Assumes number of outcomes in one experiment doesn't depend on the outcome of the other experiment.

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    - Tota: $11 \cdot 10 \cdot 9 = 990$ possibilities

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    - By the multiplication rule, there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange them.
- **Binomial coefficient**: number of subsets of size k in a group of n objects:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

The lady tasting tea

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  - Prepare 8 cups of tea, 4 milk-first, 4 tea-first
  - Present cups to friend in a **random** order
  - Ask friend to pick which 4 of the 8 were milk-first.

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- Chances of guessing all 4 correct is $\frac{1}{70} \approx 0.014$ or 1.4%.
- $\rightsquigarrow$ the guessing hypothesis might be implausible.

Birthday problem

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There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (no leap babies) and birthdays are independent. What is the probability that at least one pair of people have the same birthday?

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2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Additivity) If a series of events, $A_1, A_2, \ldots$, are disjoint, then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

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    - Probability function $\mathbb{P}$ mapping events $A \subseteq \Omega$ onto the real line.
- The function $\mathbb{P}$ must satisfy the following axioms:

1. (Non-negativity) $\mathbb{P}(A) \geq 0$ for every event A
2. (Normalization) $\mathbb{P}(\Omega) = 1$
3. (Additivity) If a series of events, $A_1, A_2, \ldots$, are disjoint, then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

- Probability function assigns "mass" to regions of the sample space.

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    - This class: focus on frequentist perspectives because it's pervasive.

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Appendix

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- An event: picking a Queen, $\{Q\clubsuit, Q\spadesuit, Q\heartsuit, Q\diamondsuit\}$

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- We want to know or model the probability of these events!

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