2: Conditional Probability

Spring 2023

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Gov 2002 (Harvard)

Roadmap

1. Conditional Probability
2. Bayes's Rule
3. Independence

1/ Conditional Probability

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- Conditional probability is the cornerstone of quantitative social science.

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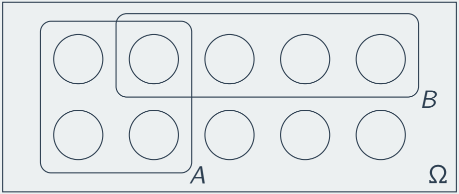
    - Also known as the **prosecutor's fallacy**

Intuition



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    - Intuitively, it's because B occurring precludes A from occurring.

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- $\rightsquigarrow$ rules of probability apply to left-hand side of conditioning bar (A)
    - All probabilities **normalized** to event B, $\mathbb{P}(B \mid B) = 1$.
- Not for right-hand side, so even if B and C are disjoint,

$$\mathbb{P}(A|B \cup C) \neq \mathbb{P}(A|B) + \mathbb{P}(A|C)$$

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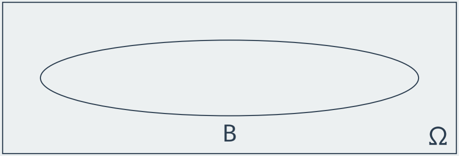
- **Actual Research Question™:** modeling the continuation probability of war, $\mathbb{P}(W_2 \mid W_1)$ and the probability of conflict resolution, $\mathbb{P}(P_3 \mid W_1, W_2)$.

Law of Total Probability



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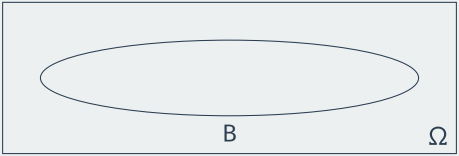

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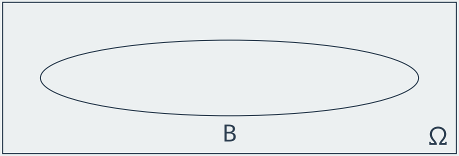

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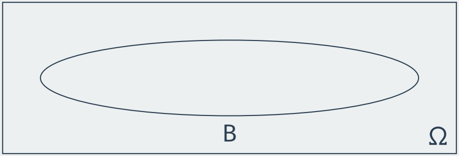

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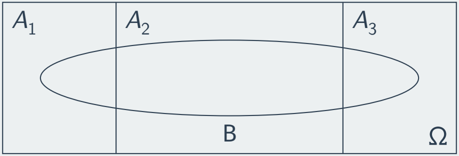

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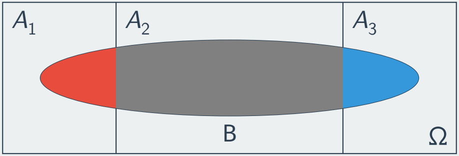

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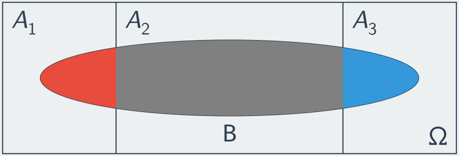

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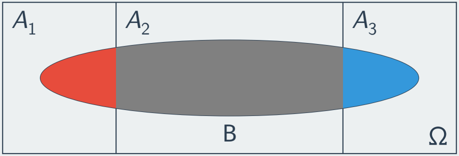

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2/ Bayes's Rule

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- **Base rate fallacy**: ignores how uncommon QAnon believers are!

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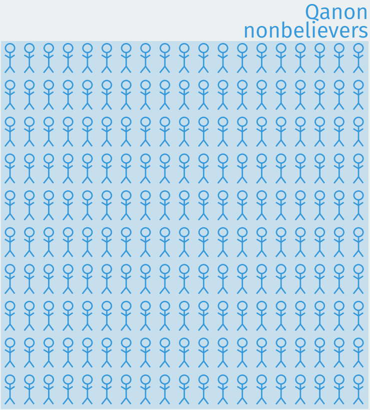

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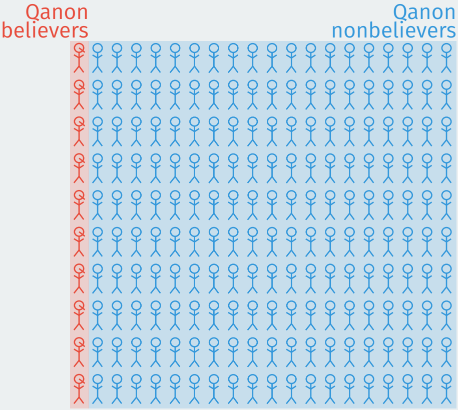

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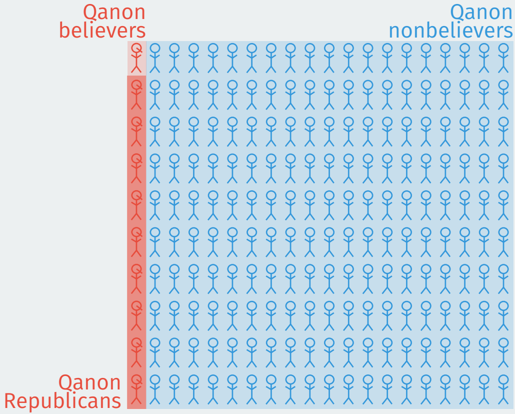

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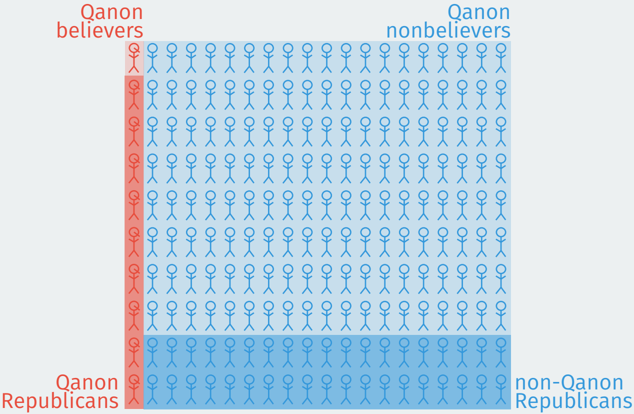

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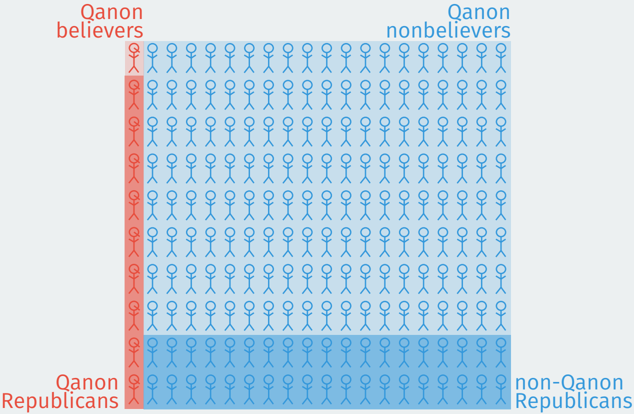

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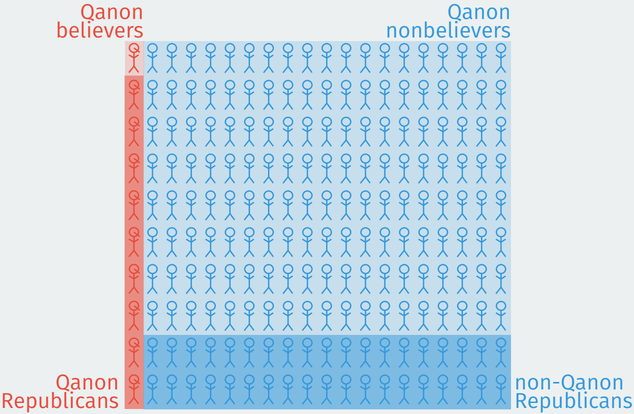

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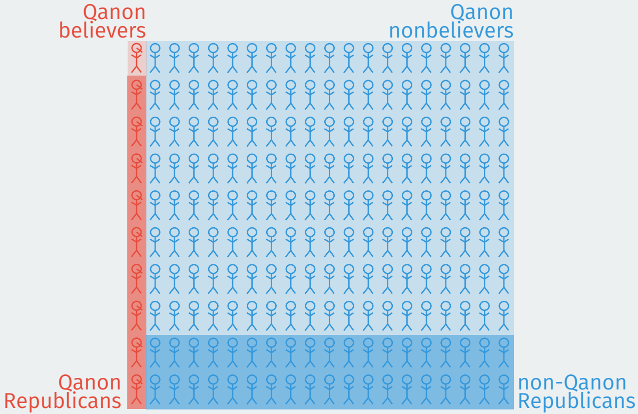

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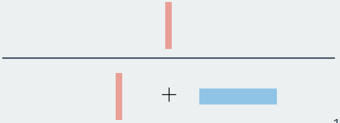

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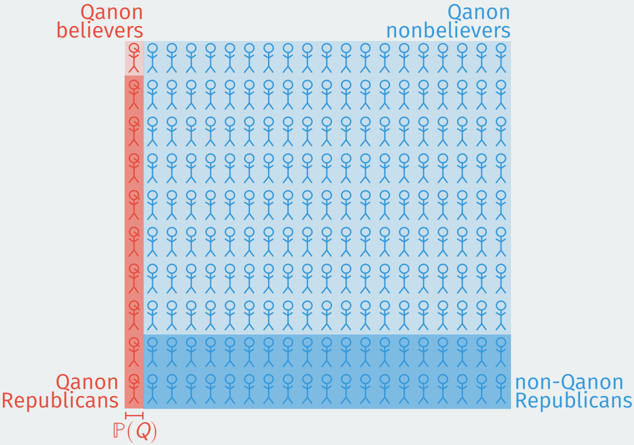
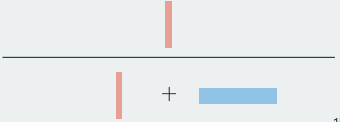


Chance a random Republican believes QAnon = 

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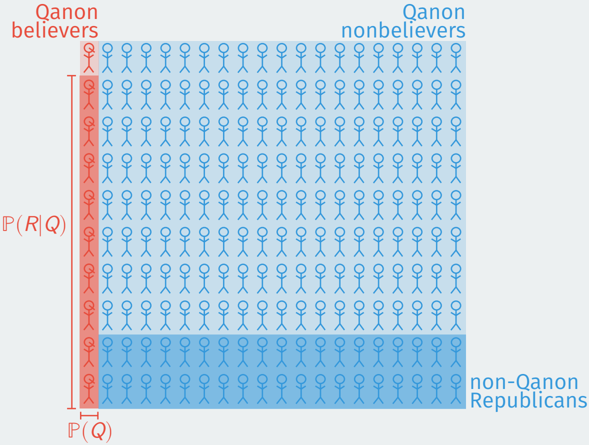

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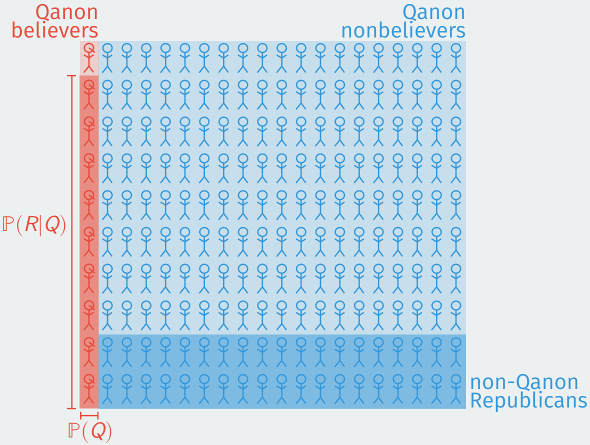

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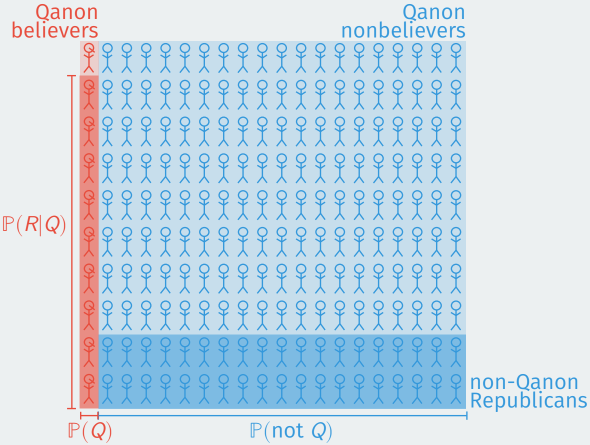

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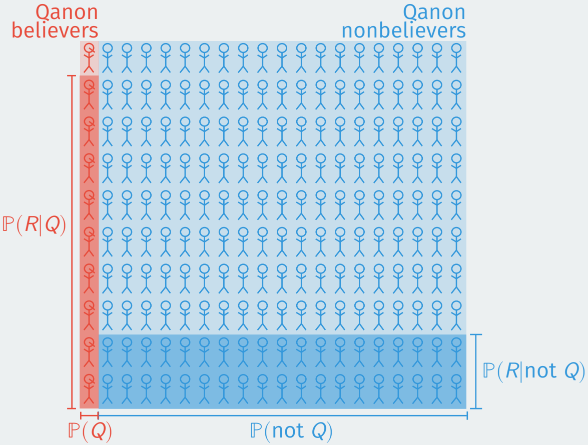

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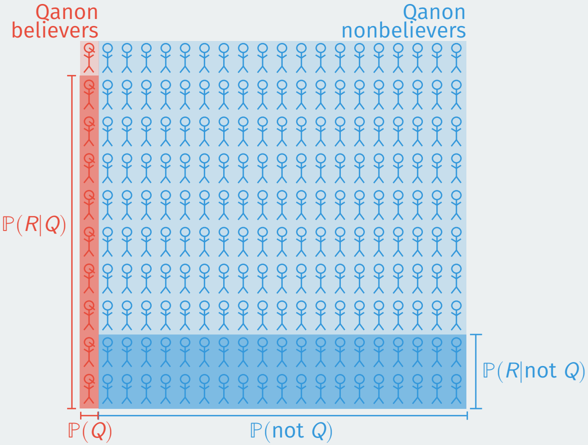

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    - $\mathbb{P}(PT|C) = 0.8$: true positive rate
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Medical tests

- Suppose you go and get a COVID-19 test and it comes back positive!
    - Let a positive test be PT.
- What's the probability you actually have COVID-19?
    - Let having COVID be labeled C.
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    - $\mathbb{P}(C) = 0.007$ rough prevalance of active COVID cases.

Applying Bayes' rule to COVID tests

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- If false positive rate goes up to 1% $\rightsquigarrow \mathbb{P}(C \mid PT) \approx 0.36$

3/ Independence

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- Common misunderstanding: **independent is different than disjoint!**
  - Mutually exclusive events provide information!

Independence example

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- When seeing "prob. of at least one" $\rightsquigarrow$ work with complement:

$$\begin{aligned}&\mathbb{P}(\text{At least one COVID case at gathering})\\ &\quad = 1 - \mathbb{P}(\text{No COVID cases at gathering})\end{aligned}$$

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  - $n = 100 \rightsquigarrow$ prob of 0.5

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    - Ind. $\not\Rightarrow$ cond. ind.: test scores, athletics, and college admission.