5: Continuous Random Variables

Spring 2023

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Gov 2002 (Harvard)

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- Learned how to define distributions (p.m.f., c.d.f.) and how to summarize.
- Now: define the same ideas for r.v.s that can take on any real value.

1/ Continuous distributions

Continuous r.v.s

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- But $\mathbb{P}(X \in (0, 1))$ must be less than 1! $\rightsquigarrow \mathbb{P}(X = x)$ must be 0.

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3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164
0628620899 8628034825 3421170679 8214808651 3282306647 0938446095 5058223172
5359408128 4811174502 8410270193 8521105559 6446229489 5493038196 4428810975
6659334461 2847564823 3786783165 2712019091 4564856692 3460348610 4543266482
1339360726 0249141273 7245870066 0631558817 4881520920 9628292540 9171536436
7892590360 0113305305 4882046652 1384146951 9415116094 3305727036 5759591953
0921861173 8193261179 3105118548 0744623799 6274956735 1885752724 8912279381
8301194912 9833673362 4406566430 8602139494 6395224737 1907021798 6094370277
0539217176 2931767523 8467481846 7669405132 0005681271 4526356082 7785771342
7577896091 7363717872 1468440901 2249534301 4654958537 1050792279 6892589235
4201995611 2129021960 8640344181 5981362977 4771309960 5187072113 4999999837
2978049951 0597317328 1609631859 5024459455 3469083026 4252230825 3344685035
2619311881 7101000313 7838752886 5875332083 8142061717 7669147303 5982534904
2875546873 1159562863 8823537875 9375195778 1857780532 1712268066 1300192787
6611195909 2164201989 3809525720 1065485863 2788659361 5338182796 8230301952
0353018529 6899577362 2599413891 2497217752 8347913151 5574857242 4541506959
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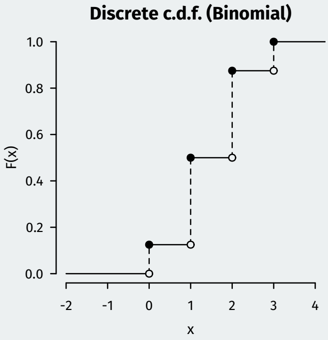
- Essentially: the c.d.f. of a continuous r.v. has no jumps:



Why "continuous"?

- How does a continuous c.d.f. connect to $\mathbb{P}(X = x)$? Note:

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- For continuous r.v.s, we'll replace the sum with an integral!

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t) dt$$

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The **probability density function** of a continuous r.v. X $f_X(x)$ is the function that satisfies

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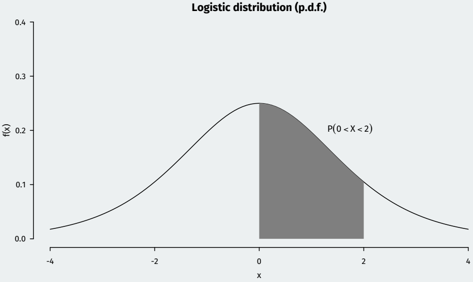

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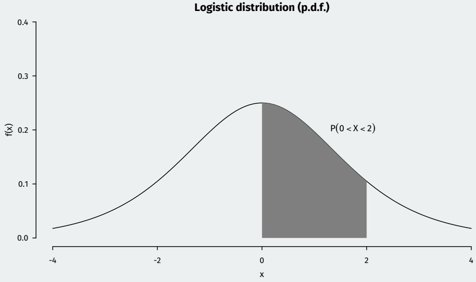

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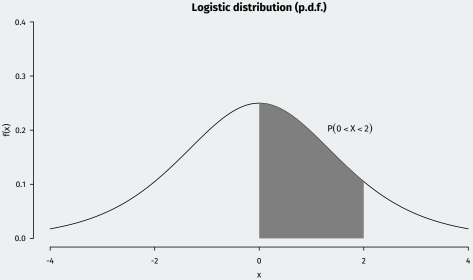

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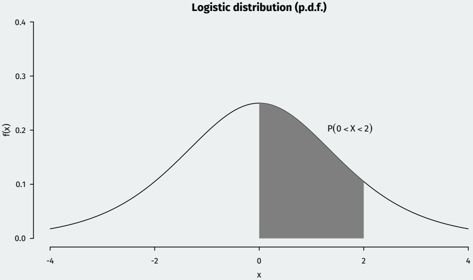

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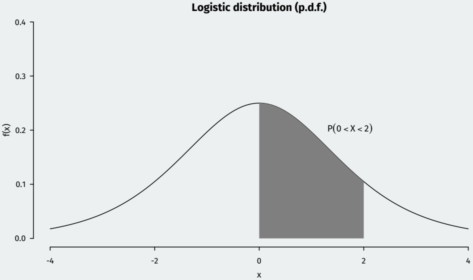

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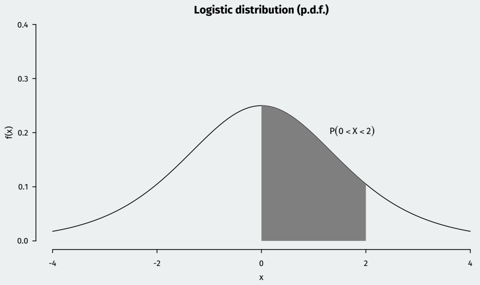

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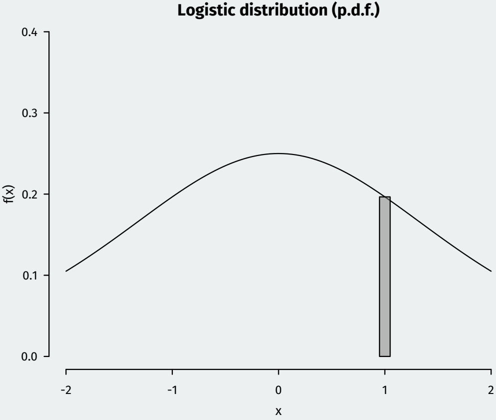
- **Important:** $f_X(x)$ can be bigger than 1!

p.d.f. intuition



- Intuition of a density:

$$f(x_0)\varepsilon \approx \mathbb{P}(X \in (x_0 - \varepsilon/2, x_0 + \varepsilon/2))$$

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$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

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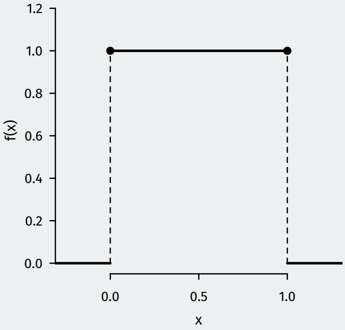
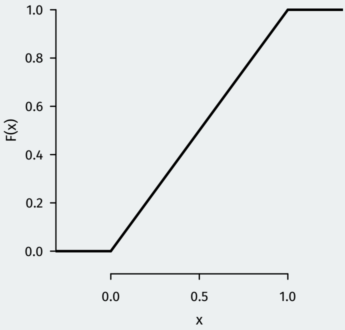
- If (c, d) is a subinterval of (a, b) then $\mathbb{P}(U \in (c, d))$ is proportional to $c - d$
- Distribution of U conditional on being in (c, d) is $\text{Unif}(c, d)$.

Uniform pdf and cdf





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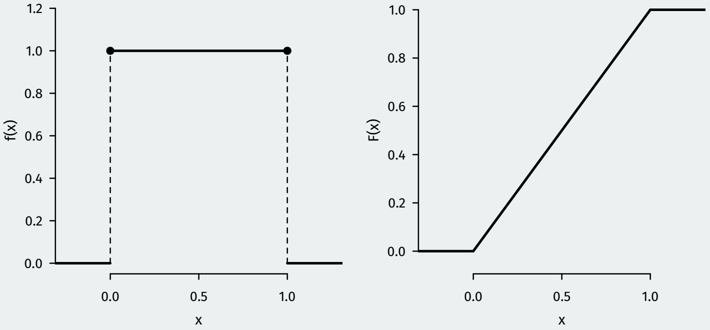

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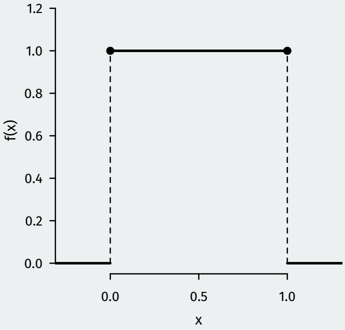

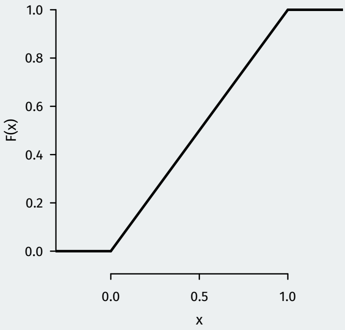

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  - Linear transformations of uniforms preserve the uniform distribution.

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  - In particular, we still have $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

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- $\rightsquigarrow \mathbb{V}[A] = 4\pi^2/45$. **Challenge:** find the c.d.f. and p.d.f. of A

3/ Universality of the uniform

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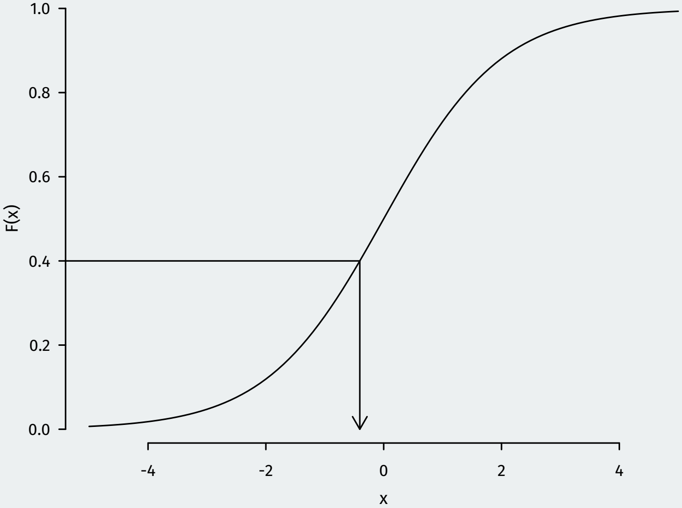
- You've probably used them before: confidence interval critical values.

Quantile functions



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    - Not $F(X) \neq \mathbb{P}(X \leq X)$.

4/ Normal distribution

Standard normal distribution

Definition

A continuous r.v. Z follows a **standard normal distribution** if its p.d.f. φ is given as

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty,$$

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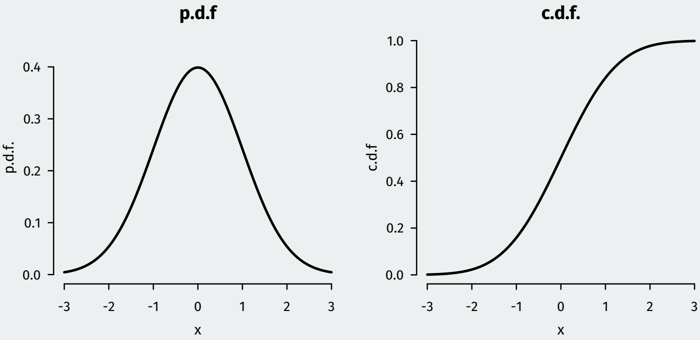
- Standard normal is mean zero, variance 1: $\mathbb{E}[Z] = 0, \mathbb{V}[Z] = 1$.

The normal distribution



- Deeply symmetric:

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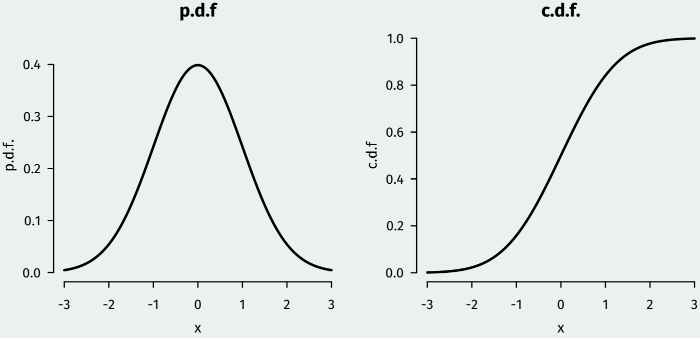

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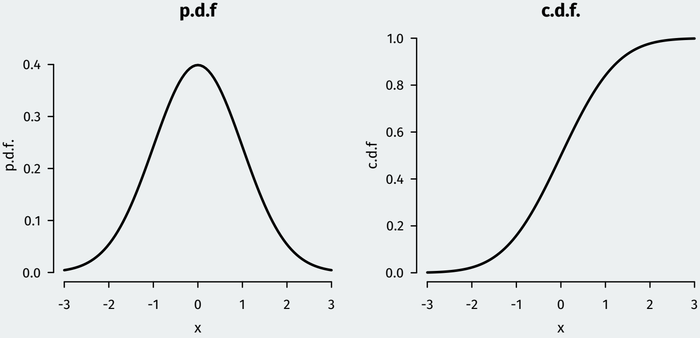

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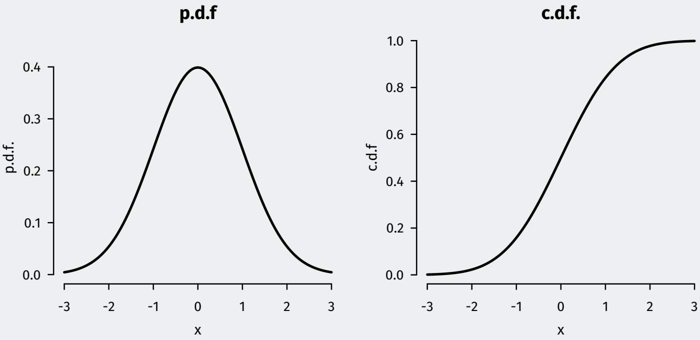

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    - Z and $-Z$ are both $\mathcal{N}(0, 1)$

General normal distribution

Defintion

If $Z \sim \mathcal{N}(0, 1)$ then

$$X = \mu + \sigma Z$$

follows the normal distribution with mean μ and variance σ^2, written $X \sim \mathcal{N}(\mu, \sigma^2)$.

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- c.d.f.: $\Phi((x - \mu)/\sigma)$
- p.d.f.:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

Properties of normals and sums

- If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and $X_1 \perp\!\!\!\perp X_2$,

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

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- **Cramer's theorem:** if $X_1 \perp\!\!\!\perp X_2$ and $X_1 + X_2$ is normal, then X_1 and X_2 are normal.

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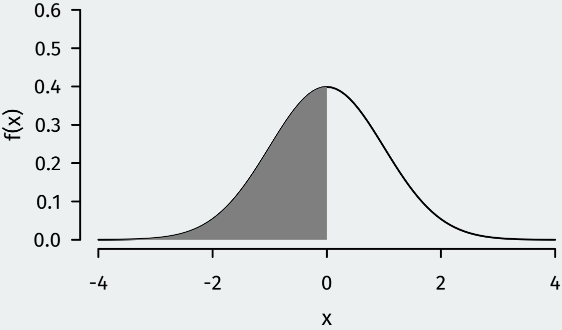


```r
pnorm(q = 0, mean = 0, sd = 1)
```

```
## [1] 0.5
```

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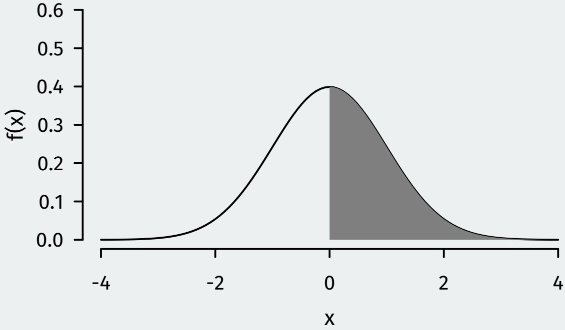


```
pnorm(q = 0, mean = 0, sd = 1, lower.tail = FALSE)
```

```
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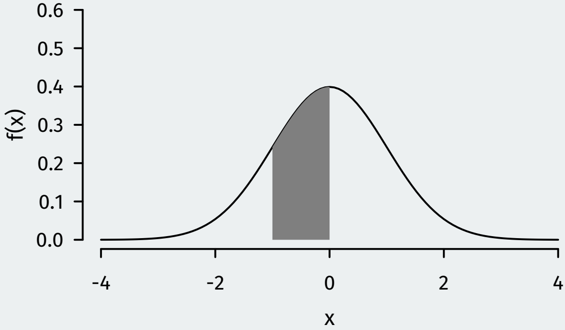
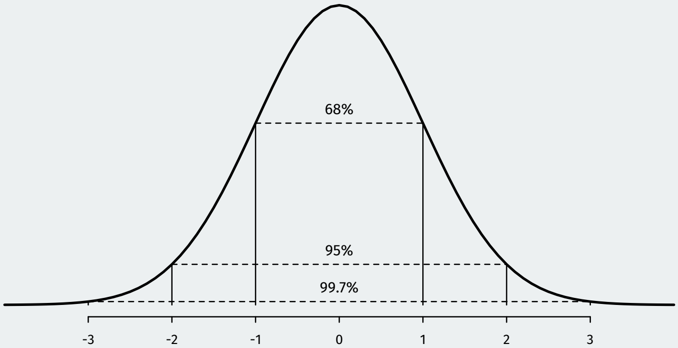



```
pnorm(q = 0, mean = 0, sd = 1) - pnorm(q = -1, mean = 0, sd = 1)
```

```
## [1] 0.341
```

Empirical Rule for the Normal Distribution



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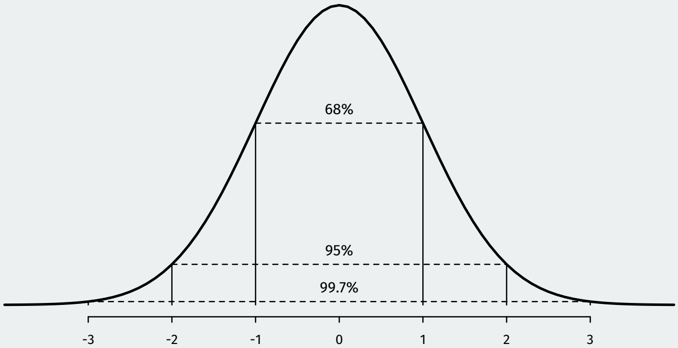

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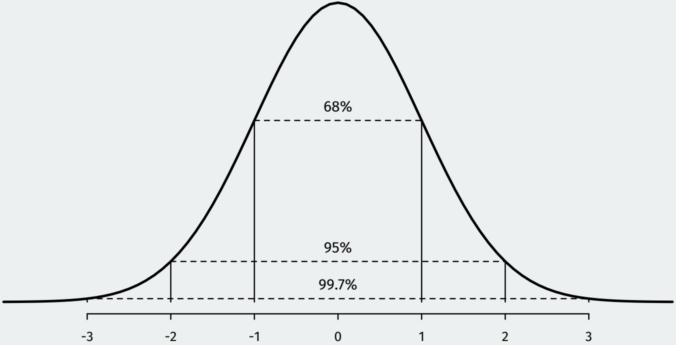

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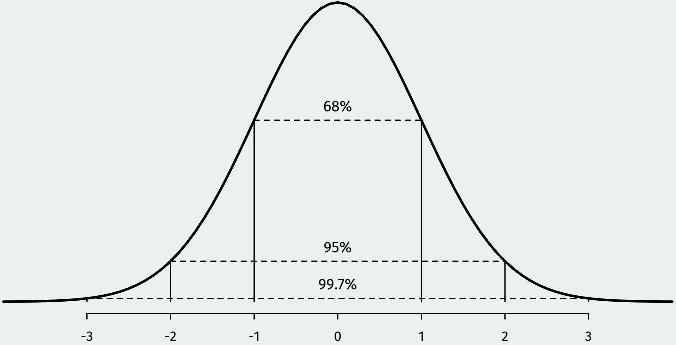

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  - Roughly 99.7% of the distribution of Z is between -3 and 3.

Chi-square distribution

Definition

Let $V = Z_1^2 + \cdots + Z_n^2$ where $Z_1, Z_2, \ldots, Z_n$ are i.i.d. $\mathcal{N}(0, 1)$. Then V follows the **Chi-square distribution** with n degrees of freedom, written $V \sim \chi_n^2$

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- Why do we care? **Sample variance** of normal r.v.s $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$:

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- Furthermore, $\overline{X}_n$ is independent of s^2/σ^2.

Student t distribution

Definition

If $Z \sim \mathcal{N}(0, 1)$ and $V \sim \chi^2_n$ with $Z \perp\!\!\!\perp V$, then

$$T = \frac{Z}{\sqrt{V/n}},$$

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  - Fatter tails than the normal.
  - Converges to $\mathcal{N}(0,1)$ as $n \to \infty$

Appendix

Symmetry of iid continuous r.v.s

Proposition

Let $X_1, \ldots, X_n$ be i.i.d. from a continuous distribution. Then,

$$\mathbb{P}(X_{a_1} < X_{a_2} < \cdots < X_{a_n}) = \frac{1}{n!}$$

for any permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$.

- All orderings of continuous i.i.d. r.v.s are equally likely.

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- Doesn't necessarily hold for discrete r.v.s