

5: Continuous Random Variables

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Gov 2002 (Harvard)

Where are we? Where are we going?

- Last few weeks: discrete random variables.
 - How to characterize uncertainty about data that takes on discrete values.
- Learned how to define distributions (p.m.f., c.d.f.) and how to summarize.
- Now: define the same ideas for r.v.s that can take on any real value.

1/ Continuous distributions

Continuous r.v.s

- Discrete r.v.: specify $\mathbb{P}(X = x)$ for all possible values \rightsquigarrow p.m.f.
- What if X can take any value on any real value?
- Can we just specify $\mathbb{P}(X = x)$ for all x ?
- No! Proof by counterexample:
 - Suppose $\mathbb{P}(X = x) = \varepsilon$ for $x \in (0, 1)$ where ε is a very small number.
 - What's the probability of being between 0 and 1?
 - There are an infinite number of real numbers between 0 and 1:

0.232879873 ... 0.57263048743 ... 0.9823612984 ...

- Each one has probability $\varepsilon \rightsquigarrow \mathbb{P}(X \in (0, 1)) = \infty \times \varepsilon = \infty$
- But $\mathbb{P}(X \in (0, 1))$ must be less than 1! $\rightsquigarrow \mathbb{P}(X = x)$ must be 0.

Thought experiment: draw a random real value between 0 and 10. What's the probability that we draw a value that is exact equal to π ?

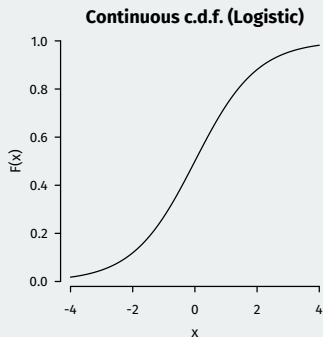
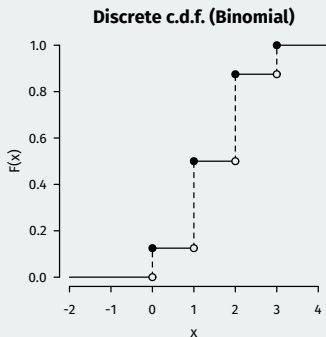
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0353018529 6899577362 2599413891 2497217752 8347913151 5574857242 4541506959
5082953311 6861727855 8890750983 8175463746 4939319255 0604009277 0167113900

Probability density functions

Definition

A r.v., X , is **continuous** if its c.d.f. $F_X(x) = \mathbb{P}(X \leq x)$ is a continuous function.

- Essentially: the c.d.f. of a continuous r.v. has no jumps:



Why “continuous”?

- How does a continuous c.d.f. connect to $\mathbb{P}(X = x)$? Note:

$$\mathbb{P}(X = x) \leq \mathbb{P}(x - \epsilon < X \leq x) = F_X(x) - F_X(x - \epsilon)$$

- But when the c.d.f. is continuous we know that

$$\mathbb{P}(X = x) \leq \lim_{\epsilon \rightarrow 0} F(x) - F(x - \epsilon) = 0$$

- Continuous c.d.f.s imply the “point probabilities” are 0. What to do?
- With discrete, we summed up the p.m.f. to get the c.d.f.

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{j: x_j \leq x} p_X(x_j)$$

- For continuous r.v.s, we'll replace the sum with an integral!

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f_X(t) dt$$

Probability density function

Definition

The **probability density function** of a continuous r.v. X $f_X(x)$ is the function that satisfies

$$F_X(x) = \int_{-\infty}^x f_X(t) dt, \quad \text{for all } x.$$

- By the fund. theorem of calculus p.d.f. is the derivative of the c.d.f.:

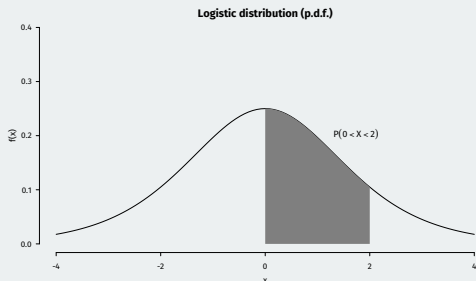
$$\frac{d}{dx} F_X(x) = f_X(x)$$

- Interval probabilities:

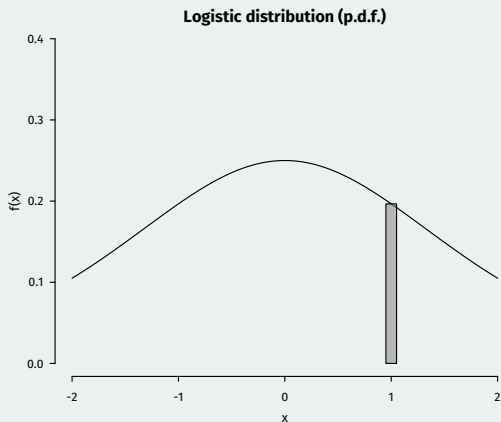
$$\mathbb{P}(a < X < b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a) = F(b) - F(a) = \int_a^b f_X(x) dx$$

- With continuous we don't have to worry about $<$ vs \leq .
 - $\mathbb{P}(a < X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b)$.

The p.d.f.



- \rightsquigarrow the probability of a region is the area under the p.d.f. for that region.
 - Support of X is all values such that $f_X(x) > 0$.
- Properties of a valid p.d.f.:
 - Nonnegative: $f_X(x) > 0$
 - Integrates to 1: $\int_{-\infty}^{\infty} f_X(x) dx = 1$
- **Important:** $f_X(x)$ can be bigger than 1!



- Intuition of a density:

$$f(x_0)\varepsilon \approx \mathbb{P}(X \in (x_0 - \varepsilon/2, x_0 + \varepsilon/2))$$

Continuous uniform distribution

- Simple and really important continuous distribution: **uniform**.
 - Intuitively, every equal-sized interval has the same probability.
 - How can figure out the p.d.f. for such a distribution?

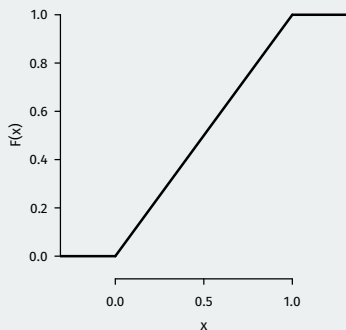
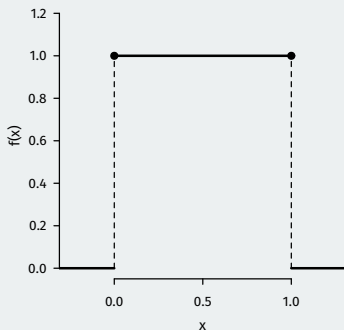
Definition

A continuous r.v. U has a **Uniform distribution** on the interval (a, b) if its p.d.f. is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

- If (c, d) is a subinterval of (a, b) then $\mathbb{P}(U \in (c, d))$ is proportional to $c - d$
- Distribution of U conditional on being in (c, d) is $\text{Unif}(c, d)$.

Uniform pdf and cdf



- **Location-scale transformation:** Let $U \sim \text{Unif}(a, b)$. Then $\tilde{U} = cU + d$ is $\text{Unif}(ca + d, cb + d)$
 - Linear transformations of uniforms preserve the uniform distribution.

2/ Expectation for continuous r.v.s

Expectation for a continuous r.v.

- Expectation of a continuous r.v.:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Unifying notation you may see: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x dF(x)$
- Expectation of a uniform (0,1): $\mathbb{E}[U] = (a + b)/2$
- LOTUS with continuous r.v.s: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
- Variance of a continuous r.v.s:

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f_X(x) dx$$

- Linearity and other properties of $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ still hold!
 - In particular, we still have $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

Expectation of random circle areas

- Let $R \sim \text{Unif}(0, 1)$ and A be the area of the circle with radius R .
- What are $\mathbb{E}[A]$ and $\mathbb{V}[A]$?
- For expectation, use LOTUS!

$$\begin{aligned}\mathbb{E}[A] &= \mathbb{E}[\pi R^2] = \int_0^1 \pi r^2 dr \\ &= (\pi/3)r^3 \Big|_0^1 \\ &= (\pi/3) \cdot 1^3 - (\pi/3) \cdot 0^3 = (\pi/3)\end{aligned}$$

- For variance, use $\mathbb{V}[A] = \mathbb{E}[A^2] - (\mathbb{E}[A])^2$:

$$\begin{aligned}\mathbb{E}[A^2] &= \mathbb{E}[\pi^2 R^4] = \int_0^1 \pi^2 r^4 dr = (\pi^2/5)r^5 \Big|_0^1 \\ &= (\pi^2/5) \cdot 1^5 - (\pi^2/5) \cdot 0^5 = (\pi^2/5)\end{aligned}$$

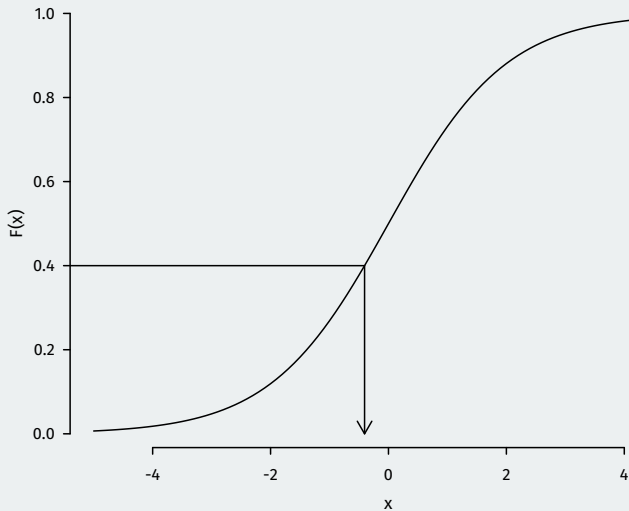
- $\rightsquigarrow \mathbb{V}[A] = 4\pi^2/45$. **Challenge:** find the c.d.f. and p.d.f. of A

3/ Universality of the uniform

Quantile function

- Inverse of the c.d.f. F^{-1} is called the **quantile function**
 - $F^{-1}(\alpha)$ is the value of x such that $\mathbb{P}(X \leq x) = \alpha$
 - Takes probabilities as arguments!
 - $F^{-1}(0.5)$ is the median, $F^{-1}(0.25)$ is the lower quartile, etc
- Intuition: exactly the same as percentiles on exams.
- You've probably used them before: confidence interval critical values.

Quantile functions



Universality of the Uniform

- The Uniform distribution has a deep connection to all continuous r.v.s
 1. Let $U \sim \text{Unif}(0, 1)$ and $X = F^{-1}(U)$, then X is an r.v. with c.d.f. F .
 2. If X is an r.v. with c.d.f. F , then $F(X) \sim \text{Unif}(0, 1)$.
- **Careful:** $F(X)$ means plug the random variable into the c.d.f. as a function.
 - Not $F(X) \neq \mathbb{P}(X \leq X)$.

4/ Normal distribution

Standard normal distribution

Definition

A continuous r.v. Z follows a **standard normal distribution** if its p.d.f. φ is given as

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty,$$

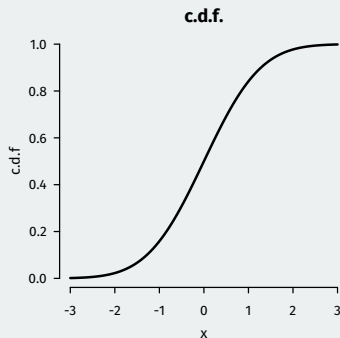
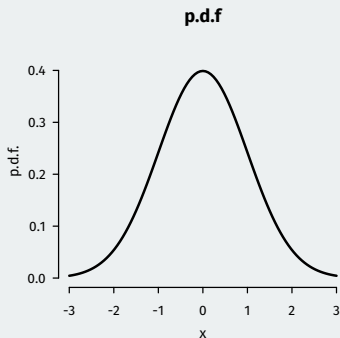
and we write this $Z \sim \mathcal{N}(0, 1)$

- Not immediately obvious, but tricky calculus will show $\int_{-\infty}^{\infty} \varphi(z) = 1$.
- Normal c.d.f. has no closed form solution, so written as:

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

- Standard normal is mean zero, variance 1: $\mathbb{E}[Z] = 0, \mathbb{V}[Z] = 1$.

The normal distribution



- Deeply symmetric:
 - p.d.f. is symmetric: $\varphi(z) = \varphi(-z)$
 - Tail areas are symmetric $\Phi(z) = 1 - \Phi(-z)$
 - Z and $-Z$ are both $\mathcal{N}(0, 1)$

General normal distribution

Definition

If $Z \sim \mathcal{N}(0, 1)$ then

$$X = \mu + \sigma Z$$

follows the normal distribution with mean μ and variance σ^2 , written $X \sim \mathcal{N}(\mu, \sigma^2)$.

- We can move back to a standard normal through **standardization**:

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1).$$

- c.d.f.: $\Phi((x - \mu)/\sigma)$
- p.d.f.:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}$$

Properties of normals and sums

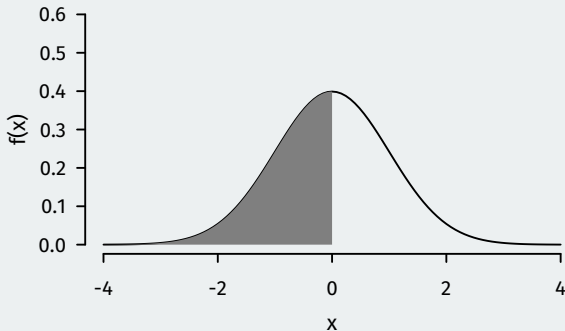
- If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and $X_1 \perp\!\!\!\perp X_2$,

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- **Cramer's theorem:** if $X_1 \perp\!\!\!\perp X_2$ and $X_1 + X_2$ is normal, then X_1 and X_2 are normal.

Using pnorm

- `pnorm()` evaluates the c.d.f. of the normal:

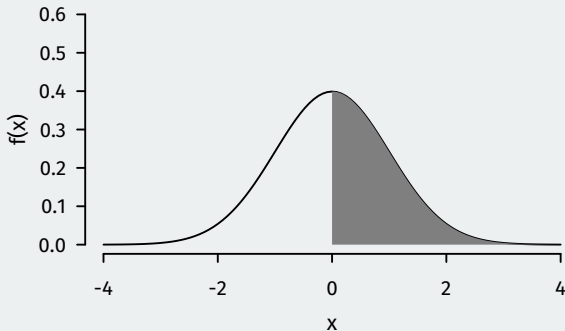


```
pnorm(q = 0, mean = 0, sd = 1)
```

```
## [1] 0.5
```

Using pnorm

- `pnorm()` evaluates the c.d.f. of the normal:

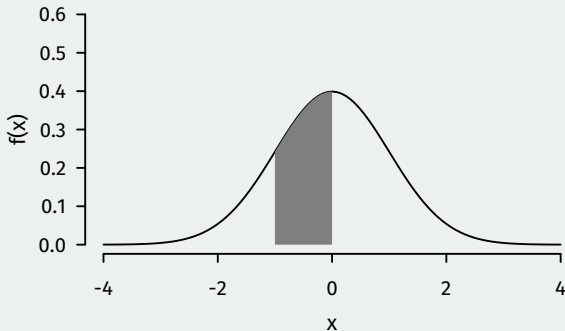


```
pnorm(q = 0, mean = 0, sd = 1, lower.tail = FALSE)
```

```
## [1] 0.5
```

Using pnorm

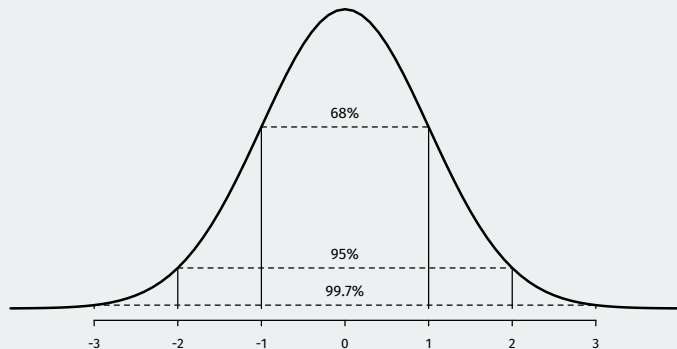
- `pnorm()` evaluates the c.d.f. of the normal:



```
pnorm(q = 0, mean = 0, sd = 1) - pnorm(q = -1, mean = 0, sd = 1)
```

```
## [1] 0.341
```

Empirical Rule for the Normal Distribution



- If $Z \sim \mathcal{N}(0, 1)$, then the following are roughly true:
 - Roughly 68% of the distribution of Z is between -1 and 1.
 - Roughly 95% of the distribution of Z is between -2 and 2.
 - Roughly 99.7% of the distribution of Z is between -3 and 3.

Chi-square distribution

Definition

Let $V = Z_1^2 + \dots + Z_n^2$ where Z_1, Z_2, \dots, Z_n are i.i.d. $\mathcal{N}(0, 1)$. Then V follows the **Chi-square distribution** with n degrees of freedom, written $V \sim \chi_n^2$

- Why do we care? **Sample variance** of normal r.v.s X_1, \dots, X_n i.i.d.

$N(\mu, \sigma^2)$:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad \frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}^2$$

- Furthermore, \bar{X}_n is independent of s^2/σ^2 .

Student t distribution

Definition

If $Z \sim \mathcal{N}(0, 1)$ and $V \sim \chi_n^2$ with $Z \perp\!\!\!\perp V$, then

$$T = \frac{Z}{\sqrt{V/n}},$$

follows the **student-t distribution** with n degrees of freedom, written $T \sim t_n$.

- Important result for the **normal model**: if X_1, \dots, X_n are i.i.d. $\mathcal{N}(\mu, \sigma^2)$:

$$T = \frac{\bar{X}_n - \mu}{\sqrt{s^2/n}} \sim t_{n-1}$$

- Properties of the t distribution:
 - Symmetric and mean-zero like the standard normal.
 - Fatter tails than the normal.
 - Converges to $\mathcal{N}(0, 1)$ as $n \rightarrow \infty$

Appendix

Symmetry of iid continuous r.v.s

Proposition

Let X_1, \dots, X_n be i.i.d. from a continuous distribution. Then,

$$\mathbb{P}(X_{a_1} < X_{a_2} < \dots < X_{a_n}) = \frac{1}{n!}$$

for any permutation a_1, a_2, \dots, a_n of $1, 2, \dots, n$.

- All orderings of continuous i.i.d. r.v.s are equally likely.
- Doesn't necessarily hold for discrete r.v.s