10. Hypothesis Testing

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Gov 2002 (Harvard)

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- Now: how to use estimates to test a particular hypothesis about a parameter.
- We'll draw on our probability knowledge from earlier in the term!

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- She guessed all correctly!
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    - There is uncertainty: she could have guessed randomly.

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- $\rightsquigarrow$ the guessing at random hypothesis might be implausible.

Social pressure effect

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY—VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

	Experimental Group				
	Control	Civic Duty	Hawthorne	Self	Neighbors
Percentage Voting	29.7%	31.5%	32.2%	34.5%	37.8%
N of Individuals	191,243	38,218	38,204	38,218	38,201

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load("../assets/gerber_green_larimer.RData")
social$voted <- 1 * (social$voted == "Yes")
neigh.mean <- mean(social$voted[social$treatment == "Neighbors"])
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- But the estimator varies from sample to sample by random chance.
- Could it be this big by random chance if there was no effect at all?

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- Estimator: sample difference in means: $\widehat{\tau}_n = \overline{Y}_{n_y} - \overline{X}_{n_x}$
- We estimate the standard error of $\widehat{\tau}_n$ with:

$$\widehat{\text{se}}[\widehat{\tau}_n] = \sqrt{\frac{s_y^2}{n_y} + \frac{s_x^2}{n_x}}$$

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  - Are traits of treatment and control groups different?

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5. Reject if T_n in rejection region, fail to reject otherwise

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  - Two-sided: $H_1 : \theta \neq \theta_0$
  - Two-sided much more common, one-sided involves ignoring evidence in one direction.

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  - Typically: larger values of $T_n \rightsquigarrow$ null less plausible.
  - A test statistic is a r.v.
- Intuitively, reject null of no effect when $|\overline{Y} - \overline{X}|$ is large.

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  - Medical diagnosis: false positive (type I) vs false negative (type II).

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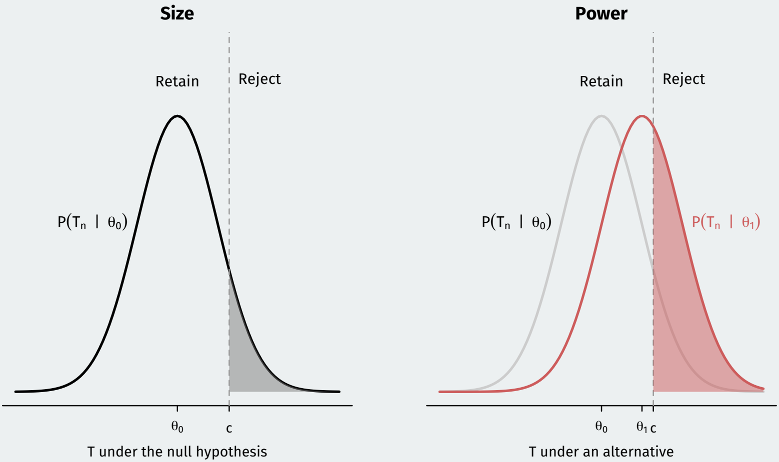
Size-power trade-off



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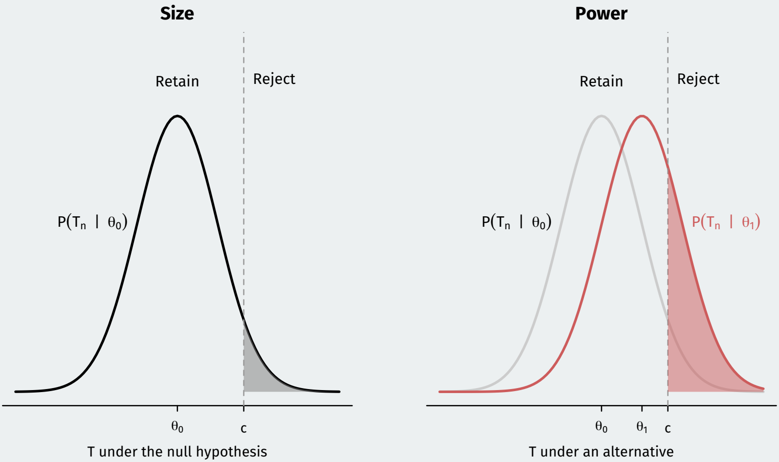

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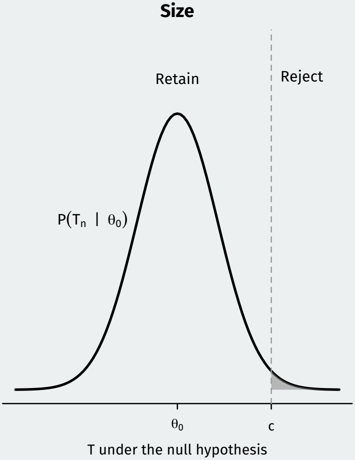

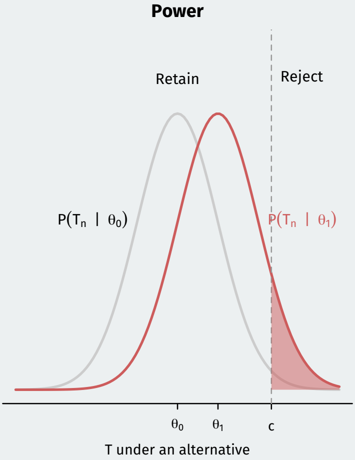

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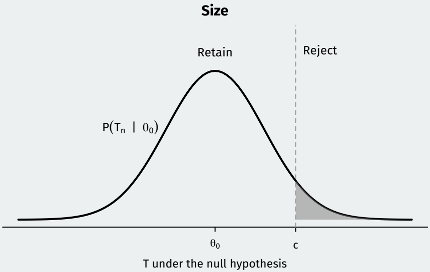

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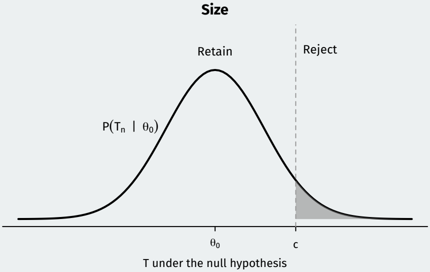

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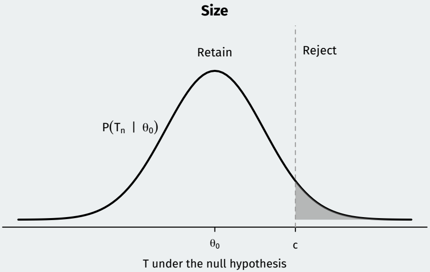

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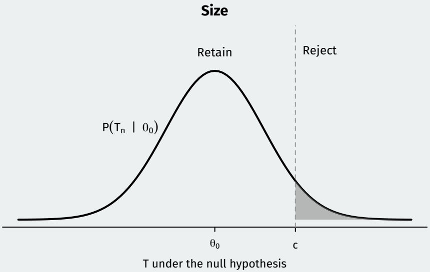

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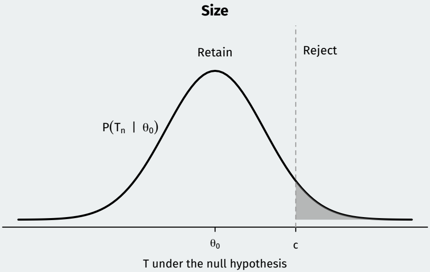

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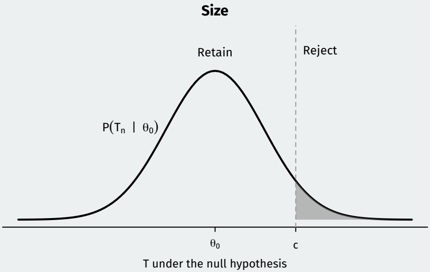

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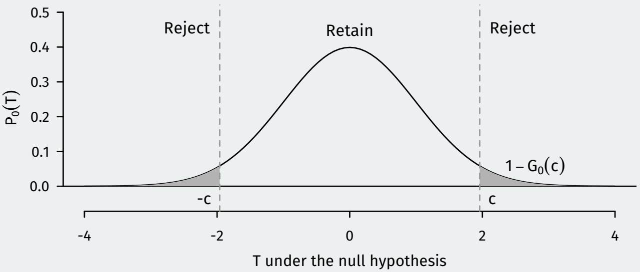
- If $G_0 \sim N(0, 1)$ and $\alpha = 0.05$, then $c = \Phi^{-1}(0.95) = 1.645$
  - Reject null if $T_n > 1.645$, fail to reject if $T_n \leq 1.645$

Two-sided rejection region



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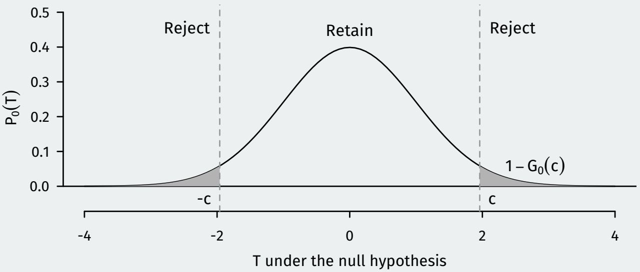

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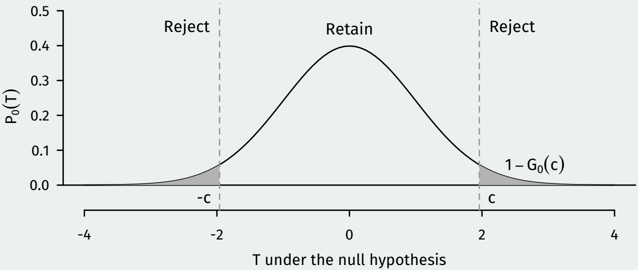

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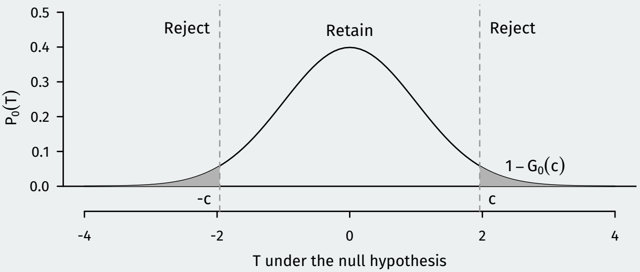

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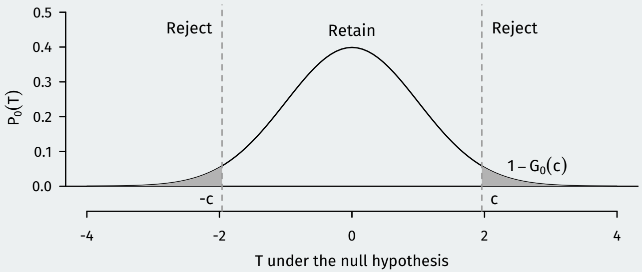

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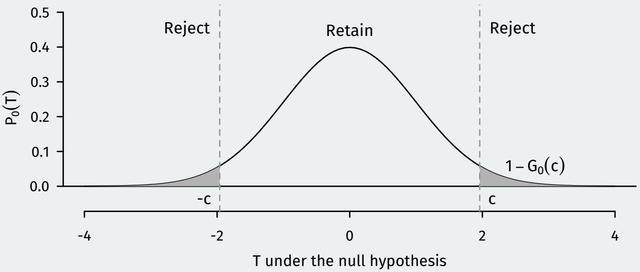

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se_diff <- sqrt(neigh_var/neigh_n + civic_var/civic_n)

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- Size of the test converges to the **nominal size** as $n \to \infty$:

$$\mathbb{P}(|T_n| > z_{\alpha/2} \mid \theta_0) \to \alpha$$

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```
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```

```
## [1] 2.06e-76
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4/ Power Analyses

Effect sizes

TABLE 2. Effects of Four Mail Treatments on Voter Turnout in the August 2006 Primary Election

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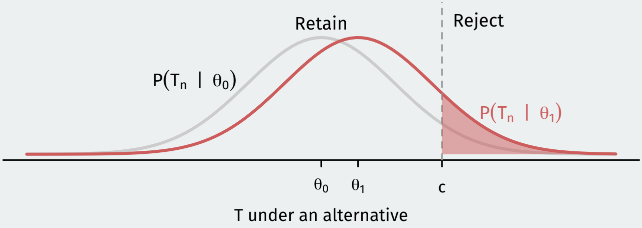
Power graph



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c
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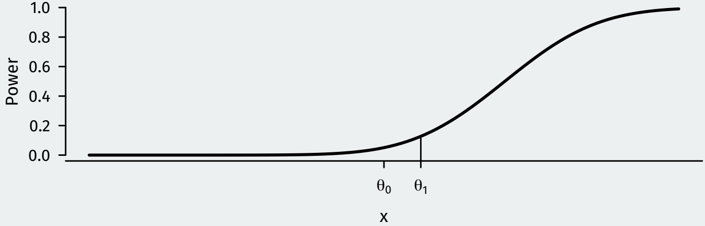

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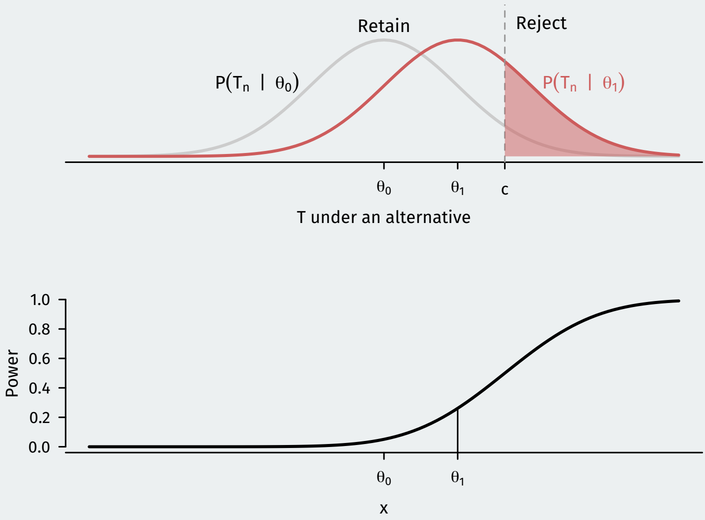
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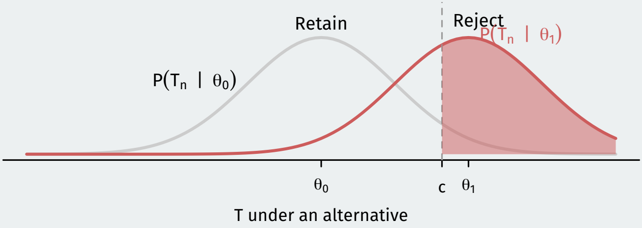

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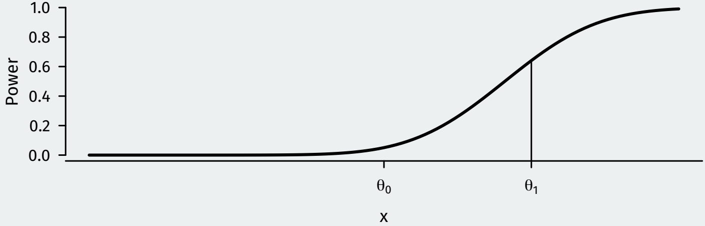

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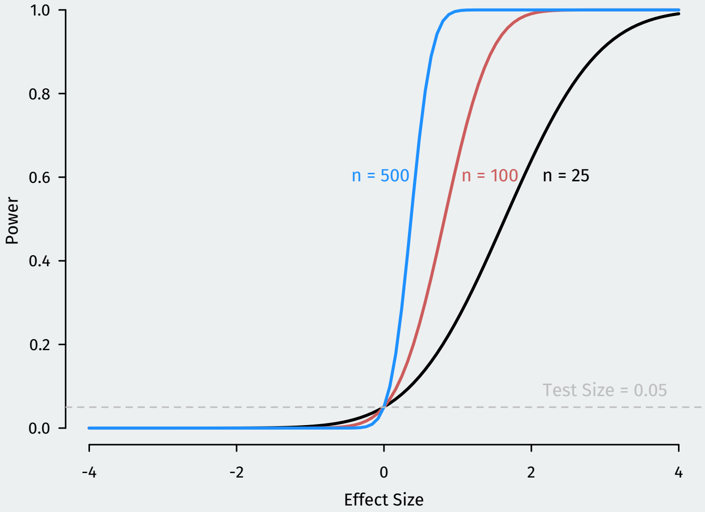

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  - Asymptotically equivalent to using the normal, but more conservative

The shape of the t