13. Properties of Least Squares

Spring 2023

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Gov 2002 (Harvard)

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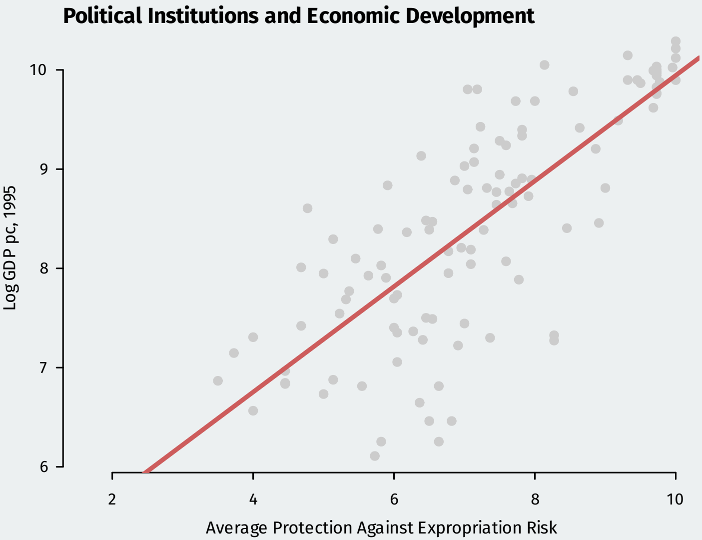
- Last time: OLS estimator, its algebraic properties.
- Now: its statistical properties, both finite-sample and asymptotic.

Acemoglu, Johnson, and Robinson (2001)



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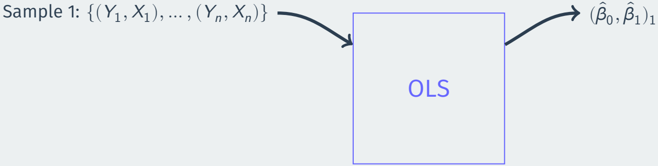

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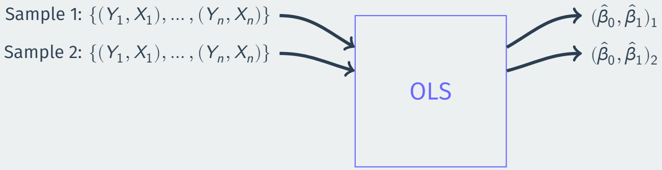

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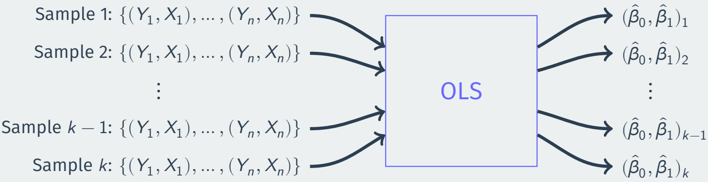

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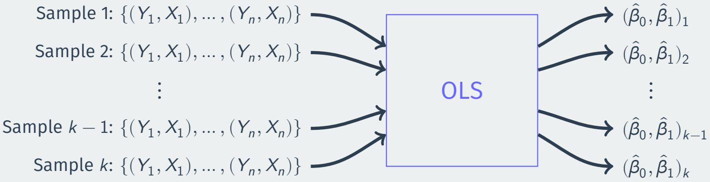

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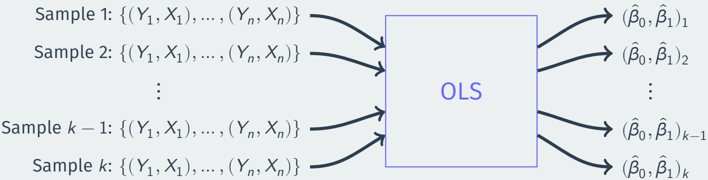

- Just like the sample mean or sample difference in means
- Has a sampling distribution, with a sampling variance/standard error.

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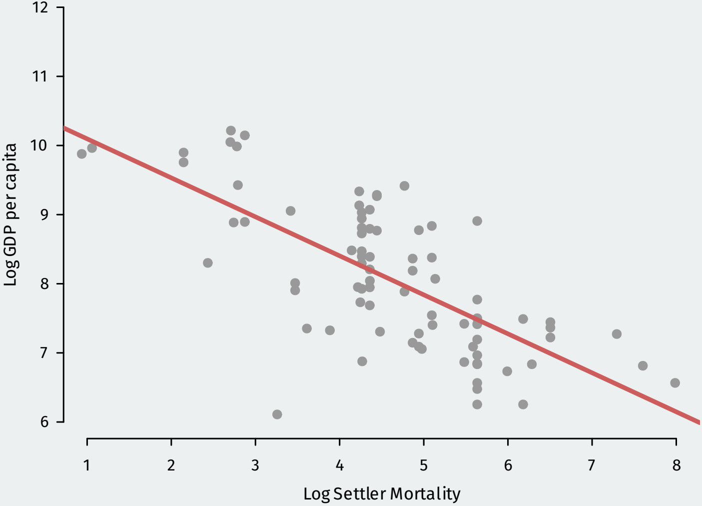
Population Regression



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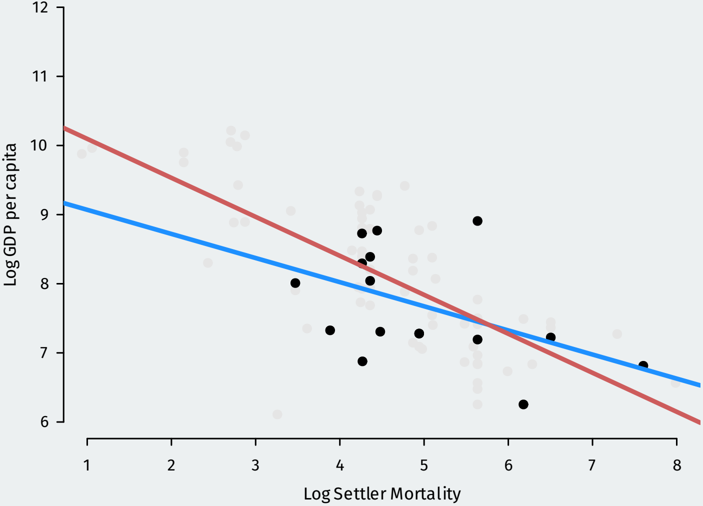

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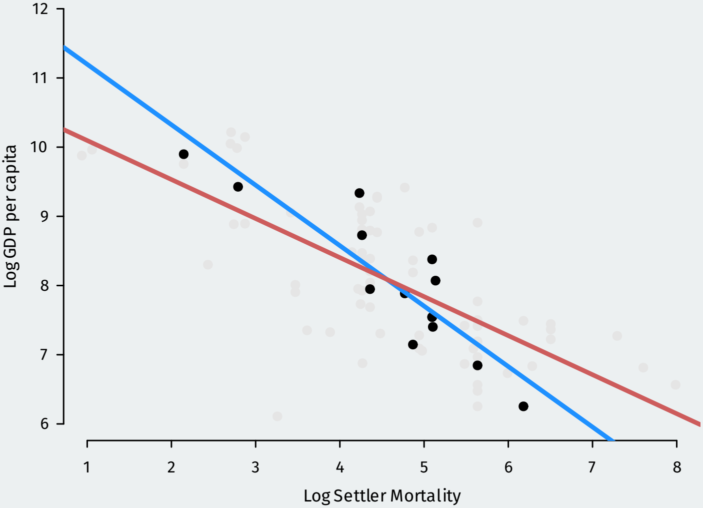

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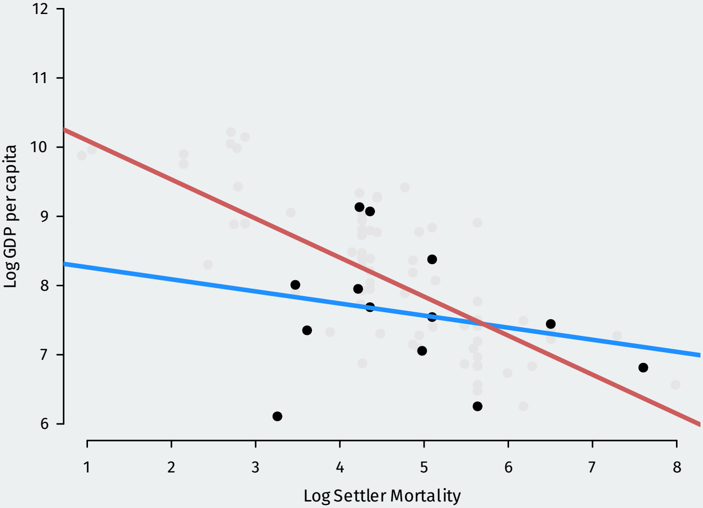

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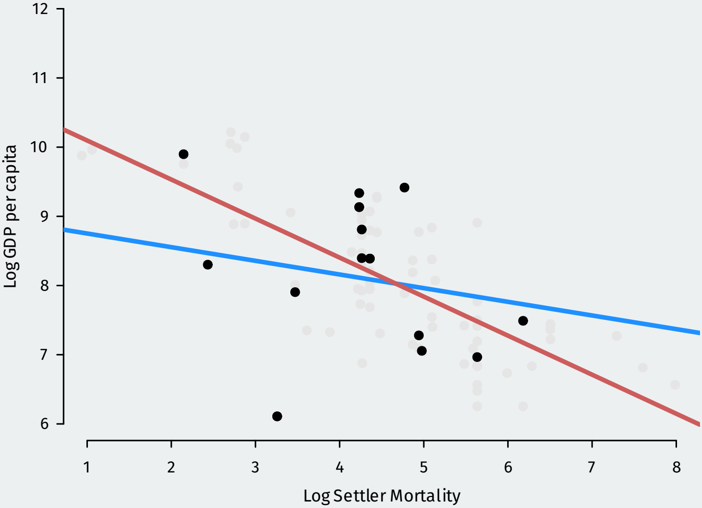

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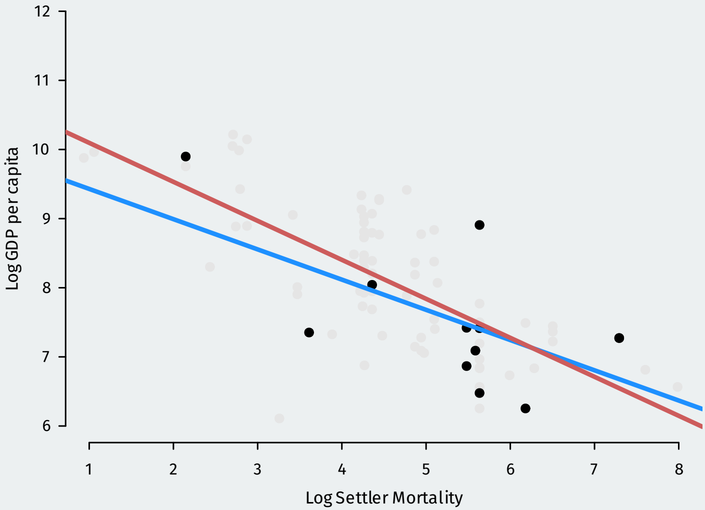

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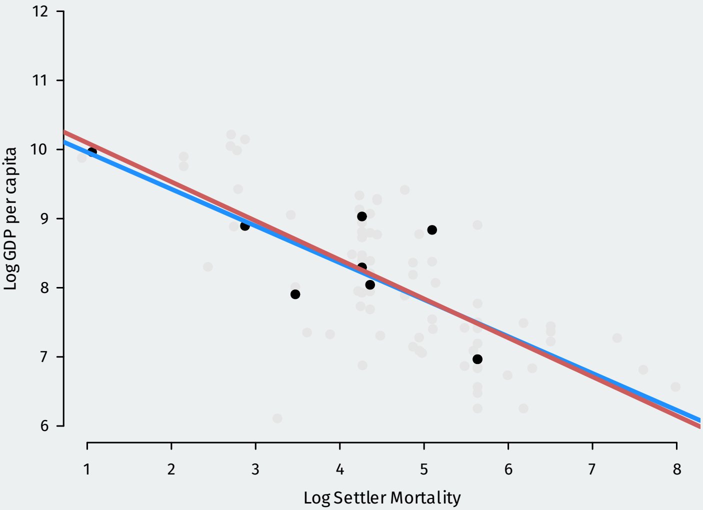

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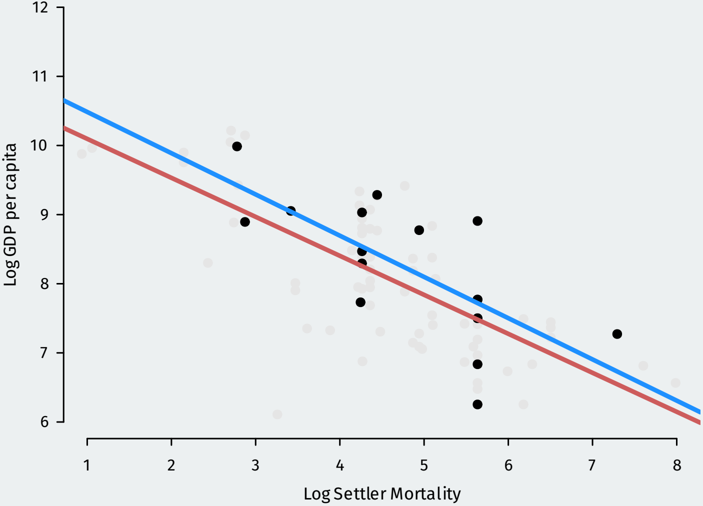

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    - **Linear regression/CEF model** for finite samples.

1/ Linear projection model and Large-sample Properties

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- What properties can we derive under such weak assumptions?

A very useful decomposition

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- $\mathbf{Q}_{\mathbf{XX}}$ is invertible by assumption, so by the continuous mapping theorem:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1} \overset{p}{\to} \mathbf{Q}_{\mathbf{XX}}^{-1} \quad\implies\quad \hat{\boldsymbol{\beta}} \overset{p}{\to} \boldsymbol{\beta} + \mathbf{Q}_{\mathbf{XX}}^{-1}\cdot\mathbf{0} = \boldsymbol{\beta},$$

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- Valid with no restrictions on Y: could be binary, discrete, etc.
- Not guaranteed to be unbiased (unless CEF is linear, as we'll see...)

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- If $\mathbb{E}[g(\mathbf{X}_i)] = 0$, then we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(\mathbf{X}_i) \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[g(\mathbf{X}_i)g(\mathbf{X}_i)'])$$

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    - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.

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$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \approx \mathbf{Q}_{\mathbf{XX}}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right)$$

- What about $n^{-1/2}\sum_{i=1}^{n}\mathbf{X}_i e_i$? Notice that:
  - $n^{-1}\sum_{i=1}^{n}\mathbf{X}_i e_i$ is a sample average with $\mathbb{E}[\mathbf{X}_i e_i] = 0$.
  - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.
- Let $\boldsymbol{\Omega} = \mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']$ and apply the CLT:

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Omega})$$

Asymptotic normality

Theorem (Asymptotic Normality of OLS)

Under the linear projection model,

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_{\boldsymbol{\beta}}),$$

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    - Square root of the diagonal of $\mathbf{V}_{\hat{\boldsymbol{\beta}}}$ = standard errors for $\hat{\beta}_j$
- Allows us to formulate (approximate) confidence intervals, tests.

2/ OLS variance estimation

Estimating OLS variance

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_{\boldsymbol{\beta}}), \qquad \mathbf{V}_{\boldsymbol{\beta}} = \mathbf{Q}_{\mathbf{XX}}^{-1}\boldsymbol{\Omega}\mathbf{Q}_{\mathbf{XX}}^{-1}$$

- Estimation of $\mathbf{V}_{\boldsymbol{\beta}}$ uses plug-in estimators.

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- Putting these together:

$$\begin{aligned}\widehat{\mathbf{V}}_{\boldsymbol{\beta}} &= \left(\frac{1}{n}\mathbb{X}'\mathbb{X}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)\left(\frac{1}{n}\mathbb{X}'\mathbb{X}\right)^{-1} \\ &= \left(\mathbb{X}'\mathbb{X}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)\left(\mathbb{X}'\mathbb{X}\right)^{-1}\end{aligned}$$

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&= \left(\mathbb{X}'\mathbb{X}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)\left(\mathbb{X}'\mathbb{X}\right)^{-1}
\end{aligned}$$

- Possible to show this is consistent: $\widehat{\mathbf{V}}_{\boldsymbol{\beta}} \xrightarrow{p} \mathbf{V}_{\boldsymbol{\beta}}$.

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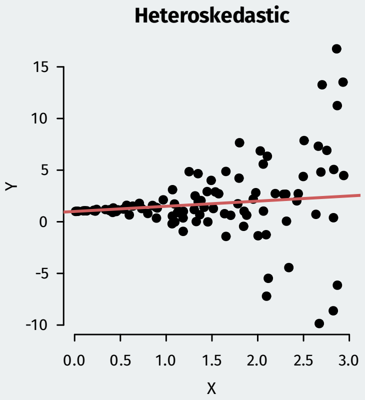
- Possible to show this is consistent: $\widehat{\mathbf{V}}_{\boldsymbol{\beta}} \xrightarrow{p} \mathbf{V}_{\boldsymbol{\beta}}$.
- Square root of the diagonal of $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}} = n^{-1}\widehat{\mathbf{V}}_{\boldsymbol{\beta}}$: **heteroskedasticity-consistent (HC) SEs** (aka “robust SEs”)

Homoskedasticity

Assumption: Homoskedasticity

The variance of the error terms is constant in $\mathbf{X}$, $\mathbb{E}[e^2 \mid \mathbf{X}] = \sigma^2(\mathbf{X}) = \sigma^2$.



Consequences of homoskedasticity

- Homoskedasticity implies $\mathbb{E}[e_i^2 \mathbf{X}_i \mathbf{X}_i'] = \mathbb{E}[e_i^2]\mathbb{E}[\mathbf{X}_i \mathbf{X}_i'] = \sigma^2 \mathbf{Q}_{\mathbf{XX}}$

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- Estimated variance of $\hat{\boldsymbol{\beta}}$ under homoskedasticity

$$s^2 = \frac{1}{n-k}\sum_{i=1}^{n} \hat{e}_i^2 \qquad \widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\mathtt{lm}} = \frac{1}{n} s^2 \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' \right)^{-1} = s^2 \left(\mathbb{X}'\mathbb{X} \right)^{-1}$$

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- LLN implies $s^2 \xrightarrow{p} \sigma^2$ and so $n\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\texttt{lm}}$ is consistent for $\mathbf{V}_{\boldsymbol{\beta}}^{\texttt{lm}}$

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- Lots of "flavors" of HC variance estimators (HC0, HC1, HC2, etc).
    - Mostly small, ad hoc changes to improve finite-sample performance.

AJR data

```r
library(sandwich)
mod <- lm(logpgp95 ~ avexpr + lat_abst + meantemp, data = ajr)
vcov(mod) ## homoskdastic V_\hat{beta}
```

```
##             (Intercept)    avexpr  lat_abst  meantemp
## (Intercept)      0.9079 -0.040952 -0.537463 -0.023246
## avexpr          -0.0410  0.004162 -0.000778  0.000605
## lat_abst        -0.5375 -0.000778  0.867588  0.016717
## meantemp        -0.0232  0.000605  0.016717  0.000705
```

```r
sandwich::vcovHC(mod, type = "HC2") ## HC2
```

```
##             (Intercept)   avexpr lat_abst  meantemp
## (Intercept)      0.9764 -0.05735 -0.29548 -0.024639
## avexpr          -0.0573  0.00538 -0.00358  0.001107
## lat_abst        -0.2955 -0.00358  0.60821  0.008792
## meantemp        -0.0246  0.00111  0.00879  0.000706
```

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$$\text{general t-statistic} = \frac{\hat{\beta}_j - b_0}{\widehat{\text{se}}(\hat{\beta}_j)} \qquad \text{“usual” t-statistic} = \frac{\hat{\beta}_j}{\widehat{\text{se}}(\hat{\beta}_j)}$$

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- Software often uses t critical values instead of normal (we’ll see why).

Inference with `lmtest::coeftest()`

```r
library(lmtest)
## homoskedastic error
lmtest::coeftest(mod)
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9528    7.27  1.2e-09 ***
## avexpr        0.4059     0.0645    6.29  5.1e-08 ***
## lat_abst     -0.1980     0.9314   -0.21    0.832
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
## HC2 variance estimator
lmtest::coeftest(mod, vcov = vcovHC(mod, type = "HC2"))
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9881    7.01  3.3e-09 ***
## avexpr        0.4059     0.0733    5.53  8.6e-07 ***
## lat_abst     -0.1980     0.7799   -0.25    0.801
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
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3/ Inference for Multiple Parameters

Inference for interactions

$$m(x, z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

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- Estimate it by plugging in the estimated coefficients: $\frac{\partial \widehat{m}(x,z)}{\partial x} = \hat{\beta}_1 + z\hat{\beta}_3$
- What if we want the variance of this effect for any value of Z?

$$\mathbb{V}\left(\frac{\partial \widehat{m}(x,z)}{\partial x}\right) = \mathbb{V}\left[\hat{\beta}_1 + z\hat{\beta}_3\right] = \mathbb{V}[\hat{\beta}_1] + z^2\mathbb{V}[\hat{\beta}_3] + 2z\text{cov}[\hat{\beta}_1, \hat{\beta}_3]$$

Inference for interactions

$$m(x, z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

- **Partial** or **marginal** effect of X at Z: $\frac{\partial m(x,z)}{\partial x} = \beta_1 + z\beta_3$
- Estimate it by plugging in the estimated coefficients: $\frac{\partial \widehat{m}(x,z)}{\partial x} = \hat{\beta}_1 + z\hat{\beta}_3$
- What if we want the variance of this effect for any value of Z?

$$\mathbb{V}\left(\frac{\partial \widehat{m}(x,z)}{\partial x}\right) = \mathbb{V}\left[\hat{\beta}_1 + z\hat{\beta}_3\right] = \mathbb{V}[\hat{\beta}_1] + z^2\mathbb{V}[\hat{\beta}_3] + 2z\text{cov}[\hat{\beta}_1, \hat{\beta}_3]$$

- Use the estimated covariance matrix:

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  - $\widehat{V}_{\hat{\beta}_1}$ is the diagonal entry of $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}$ for $\hat{\beta}_1$

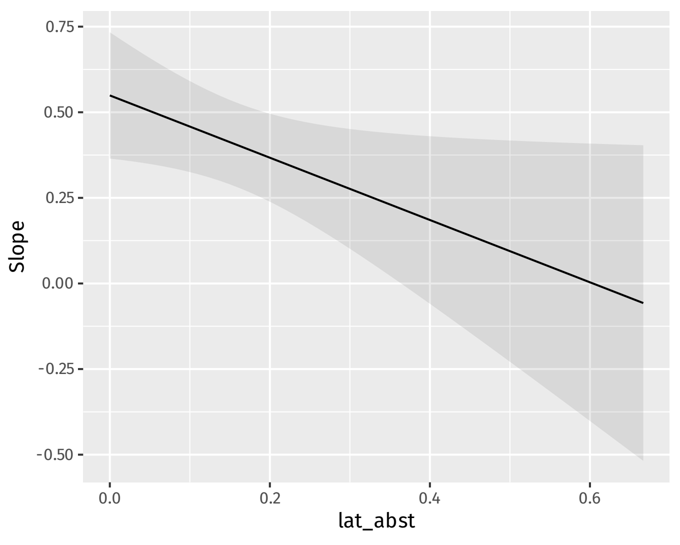
Visualizing via `marginaleffects`

```r
int_mod <- lm(logpgp95 ~ avexpr * lat_abst + meantemp, data = ajr)
coeftest(int_mod)
```

```
##
## t test of coefficients:
##
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       6.9864     0.9273    7.53    5e-10
## avexpr            0.5491     0.0941    5.84    3e-07
## lat_abst          5.8152     3.0791    1.89   0.0642
## meantemp         -0.1048     0.0326   -3.21   0.0022
## avexpr:lat_abst  -0.9095     0.4451   -2.04   0.0458
##
## (Intercept)     ***
## avexpr          ***
## lat_abst        .
## meantemp        **
## avexpr:lat_abst *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Visualizing marginal effects

```
library(marginaleffects)
plot_slopes(int_mod, variables = "avexpr", condition = "lat_abst")
```



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Alternative test for one coefficient

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Rewriting hypotheses with matrices

- We can rewrite the null hypothesis as $H_0 : \mathbf{L}\boldsymbol{\beta} = \mathbf{c}$ where,

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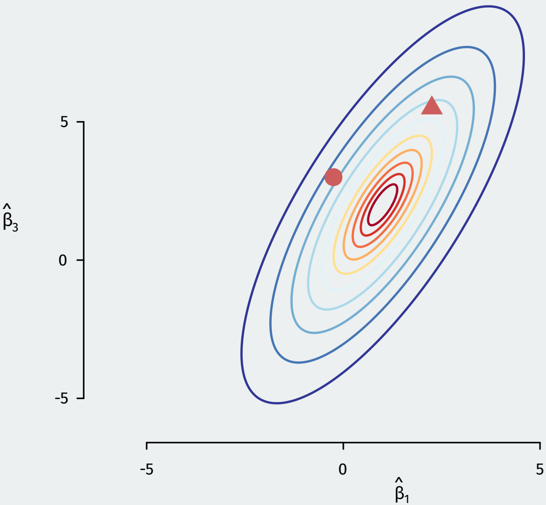
Weighting by the distribution



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- When applied to a single coefficient, equivalent to a t-test.

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    - Same interpretation: rate of false positives you are willing to accept
2. Calculate the rejection region for the test (one-sided)
    - Rejection region is the region $W > w_\alpha$ such that $\mathbb{P}(W > w_\alpha) = \alpha$
    - We can get this from R using the `qchisq()` function
3. Reject if observed statistic is bigger than critical value
    - Use `pchisq()` to get p-values if needed.

- When applied to a single coefficient, equivalent to a t-test.
- Use packages like `{lmtest}` or `{clubSandwich}` in R.

Wald test in lmtest

```r
## run OLS with the restrictions imposed (avexpr removed)
restricted <- lm(logpgp95 ~ lat_abst + meantemp, data = ajr)

## pass estimated model and estimated null model to
## wald test with HC variance estimator
lmtest::waldtest(restricted, int_mod, test = "Chisq",
                 vcov = vcovHC)
```

```
## Wald test
##
## Model 1: logpgp95 ~ lat_abst + meantemp
## Model 2: logpgp95 ~ avexpr * lat_abst + meantemp
##   Res.Df Df Chisq Pr(>Chisq)
## 1     57
## 2     55  2  34.2    3.7e-08 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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- Illustration:
    - Randomly draw 21 variables independently.
    - Run a regression of the first variable on the rest.
- By design, no effect of any variable on any other.

Multiple test example

```
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.028039   0.113820   -0.25   0.8061
## X2          -0.150390   0.112181   -1.34   0.1839
## X3           0.079158   0.095028    0.83   0.4074
## X4          -0.071742   0.104579   -0.69   0.4947
## X5           0.172078   0.114002    1.51   0.1352
## X6           0.080852   0.108341    0.75   0.4577
## X7           0.102913   0.114156    0.90   0.3701
## X8          -0.321053   0.120673   -2.66   0.0094 **
## X9          -0.053122   0.107983   -0.49   0.6241
## X10          0.180105   0.126443    1.42   0.1583
## X11          0.166386   0.110947    1.50   0.1377
## X12          0.008011   0.103766    0.08   0.9387
## X13          0.000212   0.103785    0.00   0.9984
## X14         -0.065969   0.112214   -0.59   0.5583
## X15         -0.129654   0.111575   -1.16   0.2487
## X16         -0.054446   0.125140   -0.44   0.6647
## X17          0.004335   0.112012    0.04   0.9692
## X18         -0.080796   0.109853   -0.74   0.4642
## X19         -0.085806   0.118553   -0.72   0.4713
## X20         -0.186006   0.104560   -1.78   0.0791 .
## X21          0.002111   0.108118    0.02   0.9845
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.999 on 79 degrees of freedom
## Multiple R-squared:  0.201,  Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.48
```

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    - Ensures that the family-wise error rate (probability of making at least 1 Type I error) is less than α.

4/ Linear Regression Model and Finite-sample Properties

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Assumption: Linear Regression Model

1. The variables $(Y, \mathbf{X})$ satisfy the the linear CEF assumption.

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- We continue to maintain $\{(Y_i, \mathbf{X}_i)\}$ are i.i.d.

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- Useful when linearity holds by default (discrete X in experiments, etc)

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    - A matrix $\mathbf{C}$ is p.s.d. if $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$.
    - Upshot: OLS will have the smaller SEs than any other linear estimator.

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- With reasonable n, asymptotic normality has the same effect.