13. Properties of Least Squares

Spring 2023

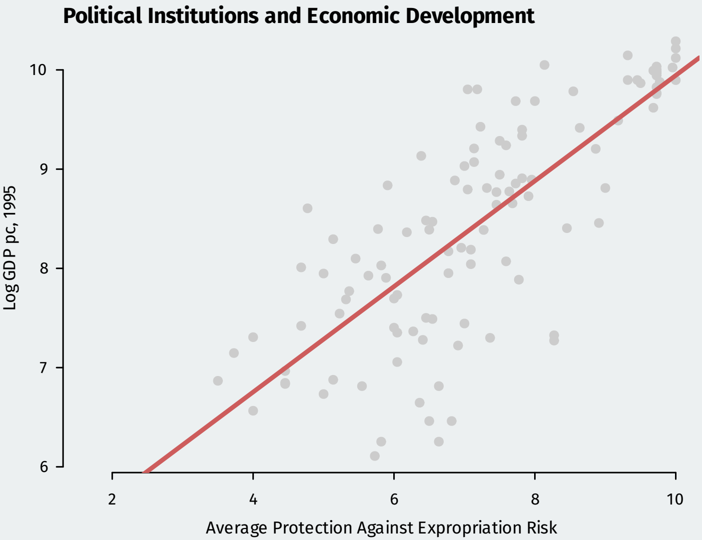
Matthew Blackwell

Gov 2002 (Harvard)

Where are we? Where are we going?

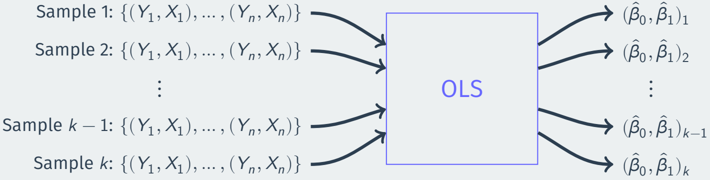
- Before: learned about CEFs and linear projections in the population.
- Last time: OLS estimator, its algebraic properties.
- Now: its statistical properties, both finite-sample and asymptotic.

Acemoglu, Johnson, and Robinson (2001)



Sampling distribution of the OLS estimator

- OLS is an estimator—we plug data into and we get out estimates.



- Just like the sample mean or sample difference in means
- Has a sampling distribution, with a sampling variance/standard error.

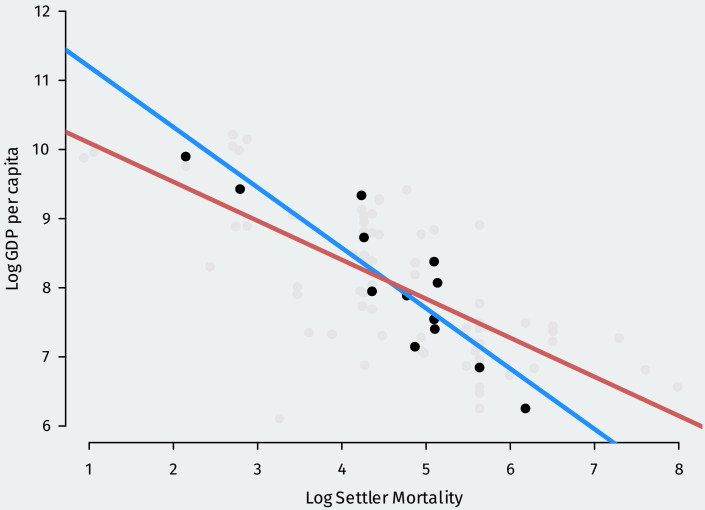
Simulation procedure

- Let's take a simulation approach to demonstrate:
    - Pretend that the AJR data represents the population of interest
    - See how the line varies from sample to sample

1. Draw a random sample of size $n = 30$ with replacement using `sample()`
2. Use `lm()` to calculate the OLS estimates of the slope and intercept
3. Plot the estimated regression line

Population Regression

Randomly sample from AJR



Big picture

- We want finite-sample guarantees about our estimates.
  - Unbiasedness, exact sampling distribution, etc.
- But finite-sample results come at a price in terms of assumptions.
  - Unbiasedness: CEF is linear.
  - Exact sampling distribution: normal errors.
- Asymptotic results hold under much weaker assumptions, but require more data.
  - OLS consistent for the linear projection even with nonlinear CEF.
  - Asymptotic normality for sampling distribution under mild assumptions.
- Focus on two models:
  - **Linear projection model** for asymptotic results.
  - **Linear regression/CEF model** for finite samples.

1/ Linear projection model and Large-sample Properties

Linear projection model

- We'll start at the most broad, fewest assumptions

Linear projection model

1. For the variables $(Y, \mathbf{X})$, we assume the linear projection of Y on $\mathbf{X}$ is defined as:
$$Y = \mathbf{X}'\boldsymbol{\beta} + e$$
$$\mathbb{E}[\mathbf{X}e] = 0.$$
2. The design matrix is invertible, so $\mathbb{E}[\mathbf{X}_i\mathbf{X}_i'] > 0$ (positive definite).

- Linear projection model holds under **very** mild assumptions.
  - Remember: not even assuming linear CEF!
  - Implies coefficients are $\boldsymbol{\beta} = (\mathbb{E}[\mathbf{X}\mathbf{X}'])^{-1}\mathbb{E}[\mathbf{X}Y]$
- What properties can we derive under such weak assumptions?

A very useful decomposition

$$\hat{\boldsymbol{\beta}} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_iY_i\right) = \boldsymbol{\beta} + \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_ie_i\right)}_{\text{estimation error}}$$

- OLS estimates are the truth plus some estimation error.
- Most of what we derive about OLS comes from this view.
- Sample means in the estimation error follow the law of large numbers:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i' \xrightarrow{p} \mathbb{E}[\mathbf{X}_i\mathbf{X}_i'] \equiv \mathbf{Q}_{\mathbf{XX}} \qquad \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_ie_i \xrightarrow{p} \mathbb{E}[\mathbf{X}e] = \mathbf{0}$$

- $\mathbf{Q}_{\mathbf{XX}}$ is invertible by assumption, so by the continuous mapping theorem:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1} \xrightarrow{p} \mathbf{Q}_{\mathbf{XX}}^{-1} \quad \Longrightarrow \quad \hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta} + \mathbf{Q}_{\mathbf{XX}}^{-1}\cdot\mathbf{0} = \boldsymbol{\beta},$$

Consistency of OLS

Theorem (Consistency of OLS)

Under the linear projection model and i.i.d. data, $\hat{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}$.

- Simple proof, but powerful result.
- OLS consistently estimates the linear projection coefficients, $\boldsymbol{\beta}$.
    - No guarantees about what the β_j represent!
    - Best linear approximation to $\mathbb{E}[Y \mid \mathbf{X}]$.
    - If we have a linear CEF, then it's consistent for the CEF coefficients.
- Valid with no restrictions on Y: could be binary, discrete, etc.
- Not guaranteed to be unbiased (unless CEF is linear, as we'll see...)

Central limit theorem, reminders

- We'll want to approximate the sampling distribution of $\hat{\boldsymbol{\beta}}$. CLT!
- Consider some sample mean of i.i.d. data: $n^{-1}\sum_{i=1}^{n} g(\mathbf{X}_i)$. We have:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right] = \mathbb{E}[g(\mathbf{X}_i)] \qquad \text{var}\left[\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right] = \frac{\text{var}[g(\mathbf{X}_i)]}{n}$$

- CLT implies:

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i) - \mathbb{E}[g(\mathbf{X}_i)]\right) \xrightarrow{d} \mathcal{N}(0, \text{var}[g(\mathbf{X}_i)])$$

- If $\mathbb{E}[g(\mathbf{X}_i)] = 0$, then we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(\mathbf{X}_i) \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[g(\mathbf{X}_i)g(\mathbf{X}_i)'])$$

Standardized estimator

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right)$$

- Remember that $(n^{-1}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i')^{-1} \xrightarrow{p} \mathbf{Q}_{\mathbf{XX}}^{-1}$ so we have

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \approx \mathbf{Q}_{\mathbf{XX}}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right)$$

- What about $n^{-1/2}\sum_{i=1}^{n}\mathbf{X}_i e_i$? Notice that:
    - $n^{-1}\sum_{i=1}^{n}\mathbf{X}_i e_i$ is a sample average with $\mathbb{E}[\mathbf{X}_i e_i] = 0$.
    - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.
- Let $\boldsymbol{\Omega} = \mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']$ and apply the CLT:

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Omega})$$

Asymptotic normality

Theorem (Asymptotic Normality of OLS)

Under the linear projection model,

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{V}_{\boldsymbol{\beta}}),$$

where,

$$\mathbf{V}_{\boldsymbol{\beta}} = \mathbf{Q}_{\mathbf{XX}}^{-1}\boldsymbol{\Omega}\mathbf{Q}_{\mathbf{XX}}^{-1} = \left(\mathbb{E}[\mathbf{X}_i\mathbf{X}_i']\right)^{-1}\mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']\left(\mathbb{E}[\mathbf{X}_i\mathbf{X}_i']\right)^{-1}$$

- $\hat{\boldsymbol{\beta}}$ is approximately normal with mean $\boldsymbol{\beta}$ and variance $\mathbf{Q}_{\mathbf{XX}}^{-1}\boldsymbol{\Omega}\mathbf{Q}_{\mathbf{XX}}^{-1}/n$
- $\mathbf{V}_{\hat{\boldsymbol{\beta}}} = \mathbf{V}_{\boldsymbol{\beta}}/n$ is the **asymptotic covariance matrix** of $\hat{\boldsymbol{\beta}}$
    - Square root of the diagonal of $\mathbf{V}_{\hat{\boldsymbol{\beta}}}$ = standard errors for $\hat{\beta}_j$
- Allows us to formulate (approximate) confidence intervals, tests.

2/ OLS variance estimation

Estimating OLS variance

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \overset{d}{\to} \mathcal{N}(0, \mathbf{V}_{\boldsymbol{\beta}}), \qquad \mathbf{V}_{\boldsymbol{\beta}} = \mathbf{Q}_{\mathbf{XX}}^{-1}\boldsymbol{\Omega}\mathbf{Q}_{\mathbf{XX}}^{-1}$$

- Estimation of $\mathbf{V}_{\boldsymbol{\beta}}$ uses plug-in estimators.
  - Replace $\mathbf{Q}_{\mathbf{XX}} = \mathbb{E}[\mathbf{X}_i\mathbf{X}_i']$ with $\widehat{\mathbf{Q}}_{\mathbf{XX}} = n^{-1}\sum_{i=1}^n \mathbf{X}_i\mathbf{X}_i' = \mathbb{X}'\mathbb{X}/n$.
  - Replace $\boldsymbol{\Omega} = \mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']$ with $\widehat{\boldsymbol{\Omega}} = n^{-1}\sum_{i=1}^n \hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'$
- Putting these together:

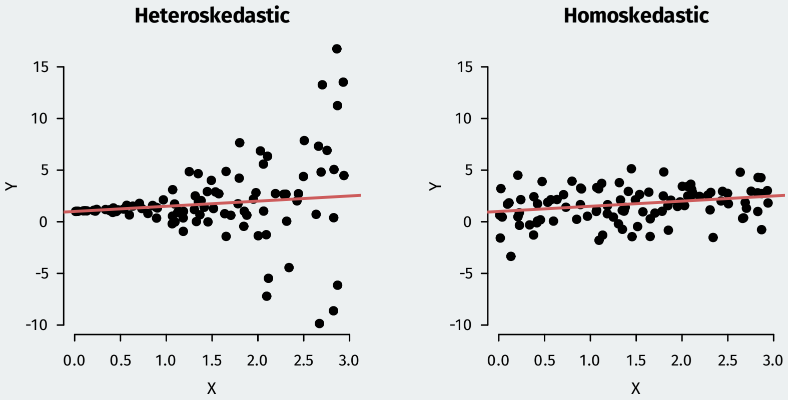
$$\begin{aligned}\widehat{\mathbf{V}}_{\boldsymbol{\beta}} &= \left(\frac{1}{n}\mathbb{X}'\mathbb{X}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)\left(\frac{1}{n}\mathbb{X}'\mathbb{X}\right)^{-1} \\ &= (\mathbb{X}'\mathbb{X})^{-1}\left(\frac{1}{n}\sum_{i=1}^n \hat{e}_i^2\mathbf{X}_i\mathbf{X}_i'\right)(\mathbb{X}'\mathbb{X})^{-1}\end{aligned}$$

- Possible to show this is consistent: $\widehat{\mathbf{V}}_{\boldsymbol{\beta}} \overset{p}{\to} \mathbf{V}_{\boldsymbol{\beta}}$.
- Square root of the diagonal of $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}} = n^{-1}\widehat{\mathbf{V}}_{\boldsymbol{\beta}}$: **heteroskedasticity-consistent (HC) SEs** (aka "robust SEs")

Homoskedasticity

Assumption: Homoskedasticity

The variance of the error terms is constant in $\mathbf{X}$, $\mathbb{E}[e^2 \mid \mathbf{X}] = \sigma^2(\mathbf{X}) = \sigma^2$.



Consequences of homoskedasticity

- Homoskedasticity implies $\mathbb{E}[e_i^2 \mathbf{X}_i \mathbf{X}_i'] = \mathbb{E}[e_i^2]\mathbb{E}[\mathbf{X}_i \mathbf{X}_i'] = \sigma^2 \mathbf{Q}_{\mathbf{XX}}$
- Simplifies the expression for the variance of $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$:

$$\mathbf{V}_{\boldsymbol{\beta}}^{\texttt{lm}} = \mathbf{Q}_{\mathbf{XX}}^{-1}\mathbb{E}[e_i^2]\mathbf{Q}_{\mathbf{XX}}\mathbf{Q}_{\mathbf{XX}}^{-1} = \sigma^2 \mathbf{Q}_{\mathbf{XX}}^{-1}$$

- Estimated variance of $\hat{\boldsymbol{\beta}}$ under homoskedasticity

$$s^2 = \frac{1}{n-k}\sum_{i=1}^{n} \hat{e}_i^2 \qquad \widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\texttt{lm}} = \frac{1}{n} s^2 \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' \right)^{-1} = s^2 \left(\mathbb{X}'\mathbb{X} \right)^{-1}$$

- LLN implies $s^2 \xrightarrow{p} \sigma^2$ and so $n\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\texttt{lm}}$ is consistent for $\mathbf{V}_{\boldsymbol{\beta}}^{\texttt{lm}}$

Notes on skedasticity

- Homoskedasticity: strong assumption that isn't needed for consistency.
- Software: almost always reports $\widehat{\mathbf{V}}^{\texttt{lm}}_{\hat{\boldsymbol{\beta}}}$ by default.
    - e.g. `lm()` in R or `reg` in Stata.
- Separate commands for HC SEs $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}$
    - Use **`{sandwich}`** package in R or **`,robust`** in Stata.
- If $\widehat{\mathbf{V}}^{\texttt{lm}}_{\hat{\boldsymbol{\beta}}}$ and $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}$ differ a lot, maybe check modeling assumptions (King and Roberts, PA 2015)
- Lots of "flavors" of HC variance estimators (HC0, HC1, HC2, etc).
    - Mostly small, ad hoc changes to improve finite-sample performance.

AJR data

```r
library(sandwich)
mod <- lm(logpgp95 ~ avexpr + lat_abst + meantemp, data = ajr)
vcov(mod) ## homoskdastic V_\hat{beta}
```

```
##             (Intercept)    avexpr  lat_abst  meantemp
## (Intercept)      0.9079 -0.040952 -0.537463 -0.023246
## avexpr          -0.0410  0.004162 -0.000778  0.000605
## lat_abst        -0.5375 -0.000778  0.867588  0.016717
## meantemp        -0.0232  0.000605  0.016717  0.000705
```

```r
sandwich::vcovHC(mod, type = "HC2") ## HC2
```

```
##             (Intercept)   avexpr lat_abst  meantemp
## (Intercept)      0.9764 -0.05735 -0.29548 -0.024639
## avexpr          -0.0573  0.00538 -0.00358  0.001107
## lat_abst        -0.2955 -0.00358  0.60821  0.008792
## meantemp        -0.0246  0.00111  0.00879  0.000706
```

Inference with OLS

- Inference is basically the same as any asymptotically normal estimator.
- Let $\widehat{\text{se}}(\hat{\beta}_j)$ be the estimated SE for $\hat{\beta}_j$.
    - Square root of jth diagonal entry: $\sqrt{[\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}]_{jj}}$
- Hypothesis test of $\beta_j = b_0$:

$$\text{general t-statistic} = \frac{\hat{\beta}_j - b_0}{\widehat{\text{se}}(\hat{\beta}_j)} \qquad \text{“usual” t-statistic} = \frac{\hat{\beta}_j}{\widehat{\text{se}}(\hat{\beta}_j)}$$

    - Use same critical values from the normal as usual $z_{\alpha/2} = 1.96$.
- 95% (asymptotic) confidence interval for $\hat{\beta}_j$:

$$\left[\hat{\beta}_j - 1.96\, \widehat{\text{se}}(\hat{\beta}_j),\ \hat{\beta}_j + 1.96\, \widehat{\text{se}}(\hat{\beta}_j)\right]$$

- Software often uses t critical values instead of normal (we’ll see why).

Inference with `lmtest::coeftest()`

```r
library(lmtest)
## homoskedastic error
lmtest::coeftest(mod)
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9528    7.27  1.2e-09 ***
## avexpr        0.4059     0.0645    6.29  5.1e-08 ***
## lat_abst     -0.1980     0.9314   -0.21    0.832
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
## HC2 variance estimator
lmtest::coeftest(mod, vcov = vcovHC(mod, type = "HC2"))
```

```
##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   6.9289     0.9881    7.01  3.3e-09 ***
## avexpr        0.4059     0.0733    5.53  8.6e-07 ***
## lat_abst     -0.1980     0.7799   -0.25    0.801
## meantemp     -0.0641     0.0266   -2.41    0.019 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

3/ Inference for Multiple Parameters

Inference for interactions

$$m(x, z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

- **Partial** or **marginal** effect of X at Z: $\frac{\partial m(x,z)}{\partial x} = \beta_1 + z\beta_3$
- Estimate it by plugging in the estimated coefficients: $\frac{\partial \widehat{m}(x,z)}{\partial x} = \hat{\beta}_1 + z\hat{\beta}_3$
- What if we want the variance of this effect for any value of Z?

$$\mathbb{V}\left(\frac{\partial \widehat{m}(x,z)}{\partial x}\right) = \mathbb{V}\left[\hat{\beta}_1 + z\hat{\beta}_3\right] = \mathbb{V}[\hat{\beta}_1] + z^2\mathbb{V}[\hat{\beta}_3] + 2z\text{cov}[\hat{\beta}_1, \hat{\beta}_3]$$

- Use the estimated covariance matrix:

$$\hat{\mathbb{V}}\left(\frac{\partial \widehat{m}(x,z)}{\partial x}\right) = \widehat{V}_{\hat{\beta}_1} + z^2\widehat{V}_{\hat{\beta}_3} + 2z\widehat{V}_{\hat{\beta}_1\hat{\beta}_3}$$

  - $\widehat{V}_{\hat{\beta}_1}$ is the diagonal entry of $\widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}$ for $\hat{\beta}_1$

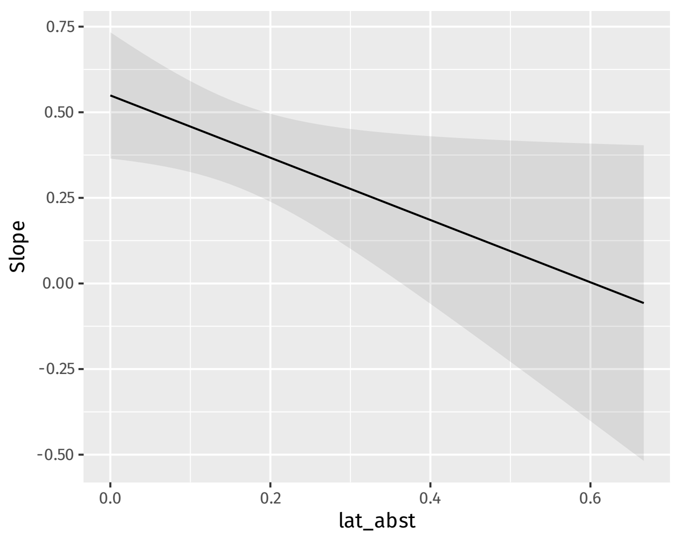
Visualizing via marginaleffects

```r
int_mod <- lm(logpgp95 ~ avexpr * lat_abst + meantemp, data = ajr)
coeftest(int_mod)
```

```
##
## t test of coefficients:
##
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       6.9864     0.9273    7.53    5e-10
## avexpr            0.5491     0.0941    5.84    3e-07
## lat_abst          5.8152     3.0791    1.89   0.0642
## meantemp         -0.1048     0.0326   -3.21   0.0022
## avexpr:lat_abst  -0.9095     0.4451   -2.04   0.0458
##
## (Intercept)     ***
## avexpr          ***
## lat_abst        .
## meantemp        **
## avexpr:lat_abst *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Visualizing marginal effects

```r
library(marginaleffects)
plot_slopes(int_mod, variables = "avexpr", condition = "lat_abst")
```



Tests of multiple coefficients

$$m(X, Z) = \beta_0 + X\beta_1 + Z\beta_2 + XZ\beta_3$$

- What about a test of no effect of X ever? Involves 2 coeffcients:

$$H_0 : \beta_1 = \beta_3 = 0$$

- Alternative: $H_1 : \beta_1 \neq 0$ or $\beta_3 \neq 0$
- We would like a test statistic that is large when the null is implausible.
  - What about $\widehat{\beta}_1^2 + \widehat{\beta}_3^2$?
  - Distribution depends on the variance/covariance of the coefficients.
  - Need to normalize like the t-statistic.

Alternative test for one coefficient

- Usually t-test of $H_0 : \beta_j = b_0$ based on the t-statistic:

$$t = \frac{\hat{\beta}_j - b_0}{\widehat{\text{se}}(\hat{\beta}_j)},$$

  - Reject when $|t| > c$ for some critical value c from the standard normal.

- Equivalent test based rejects when $t^2 > c^2$

$$t^2 = \frac{\left(\hat{\beta}_j - b_0\right)^2}{\mathbb{V}[\hat{\beta}_j]} = \frac{n\left(\hat{\beta}_j - b_0\right)^2}{[\mathbf{V}_{\boldsymbol{\beta}}]_{jj}}$$

- Because $t \overset{d}{\rightarrow} \mathcal{N}(0, 1)$, we'll have t^2 converging to a χ^2_1 distribution
  - Reminder: χ^2_k is the sum of k squared standard normals.
  - Could get the critical value for t^2 directly from χ^2_1.

Rewriting hypotheses with matrices

- We can rewrite the null hypothesis as $H_0 : \mathbf{L}\boldsymbol{\beta} = \mathbf{c}$ where,

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

    - $\mathbf{L}$ has q rows or restriction and $k+1$ columns (one for each coefficient)

- Estimated version of the constraint: $\mathbf{L}\hat{\boldsymbol{\beta}}$

- By the Delta method, under the null hypothesis we have

$$\sqrt{n}\left(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{L}\boldsymbol{\beta}\right) \overset{d}{\to} \mathcal{N}(0, \mathbf{L}'\mathbf{V}_{\boldsymbol{\beta}}\mathbf{L}).$$

- In this case:

$$\sqrt{n}\left(\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_3 \end{bmatrix}\right) \overset{d}{\to} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} [\mathbf{V}_{\boldsymbol{\beta}}]_{[11]} & [\mathbf{V}_{\boldsymbol{\beta}}]_{[13]} \\ [\mathbf{V}_{\boldsymbol{\beta}}]_{[31]} & [\mathbf{V}_{\boldsymbol{\beta}}]_{[33]} \end{bmatrix}\right)$$

    - If this covariance matrix where identity, then these would be standard normal and $\hat{\beta}_1^2 + \hat{\beta}_3^2$ would be χ_2^2 under the null

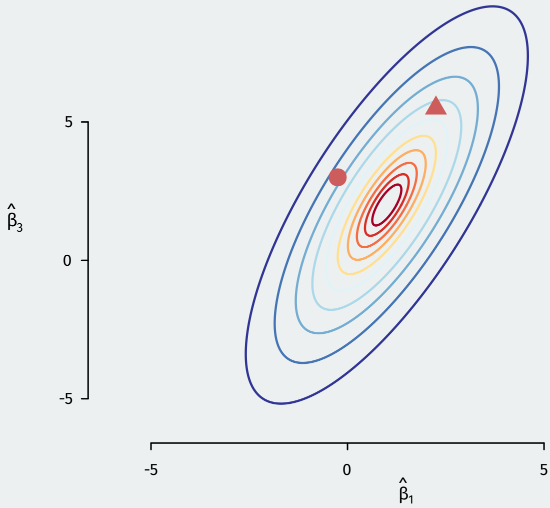
Wald statistic

- Under the null, $\sqrt{n}\left(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c}\right) \overset{d}{\to} \mathcal{N}(0, \mathbf{L}'\mathbf{V}_{\boldsymbol{\beta}}\mathbf{L})$
- $(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c})'(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c})$ is the squared deviations from the null.
  - Problem: doesn't account for variance/covariance of the estimated coefficients.
- **Wald statistic** normalize by the covariance matrix:

$$W = n\left(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c}\right)'\left(\mathbf{L}'\widehat{\mathbf{V}}_{\boldsymbol{\beta}}\mathbf{L}\right)^{-1}\left(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c}\right)$$

  - Similar to dividing by the SE for the t-test
  - Squared distance of observed values from the null, weighted by the distribution of the parameters under the null

Weighting by the distribution



Wald test

$$W = n\left(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c}\right)'\left(\mathbf{L}'\widehat{\mathbf{V}}_{\boldsymbol{\beta}}\mathbf{L}\right)^{-1}\left(\mathbf{L}\hat{\boldsymbol{\beta}} - \mathbf{c}\right)$$

- Asymptotically under the null $W \overset{d}{\to} \chi^2_q$ where q is rows of $\mathbf{L}$
  - q is the number of linear restrictions in the null
- **Wald test**: reject when $W > w_\alpha$, where $\mathbb{P}(W > w_\alpha) = \alpha$ under the null.
  - Use χ^2_q distribution for critical values, p-values
- Typical software output: **F-statistic** $F = W/q$
  - p-values and critical values come from F distribution with q and $n - k - 1$ dfs.
  - As $n \to \infty$, $F_{q,n-k-1} \overset{d}{\to} \chi^2_q$ so asymptotically similar to Wald under homoskedascity (slightly more conservative).
  - No justification for F test under heteroskedasticity.
  - "Usual" F-test reports test of all coef = 0 except intercept (pointless?)

Wald test steps

1. Choose a Type I error rate, α.
    - Same interpretation: rate of false positives you are willing to accept
2. Calculate the rejection region for the test (one-sided)
    - Rejection region is the region $W > w_\alpha$ such that $\mathbb{P}(W > w_\alpha) = \alpha$
    - We can get this from R using the `qchisq()` function
3. Reject if observed statistic is bigger than critical value
    - Use `pchisq()` to get p-values if needed.

- When applied to a single coefficient, equivalent to a t-test.
- Use packages like `{lmtest}` or `{clubSandwich}` in R.

Wald test in lmtest

```
## run OLS with the restrictions imposed (avexpr removed)
restricted <- lm(logpgp95 ~ lat_abst + meantemp, data = ajr)

## pass estimated model and estimated null model to
## wald test with HC variance estimator
lmtest::waldtest(restricted, int_mod, test = "Chisq",
                 vcov = vcovHC)
```

```
## Wald test
##
## Model 1: logpgp95 ~ lat_abst + meantemp
## Model 2: logpgp95 ~ avexpr * lat_abst + meantemp
##   Res.Df Df Chisq Pr(>Chisq)
## 1     57
## 2     55  2  34.2    3.7e-08 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Multiple testing

- Separate t-tests for each β_j: α of them will be significant by chance.
- Illustration:
    - Randomly draw 21 variables independently.
    - Run a regression of the first variable on the rest.
- By design, no effect of any variable on any other.

Multiple test example

```
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.028039   0.113820   -0.25   0.8061
## X2          -0.150390   0.112181   -1.34   0.1839
## X3           0.079158   0.095028    0.83   0.4074
## X4          -0.071742   0.104579   -0.69   0.4947
## X5           0.172078   0.114002    1.51   0.1352
## X6           0.080852   0.108341    0.75   0.4577
## X7           0.102913   0.114156    0.90   0.3701
## X8          -0.321053   0.120673   -2.66   0.0094 **
## X9          -0.053122   0.107983   -0.49   0.6241
## X10          0.180105   0.126443    1.42   0.1583
## X11          0.166386   0.110947    1.50   0.1377
## X12          0.008011   0.103766    0.08   0.9387
## X13          0.000212   0.103785    0.00   0.9984
## X14         -0.065969   0.112214   -0.59   0.5583
## X15         -0.129654   0.111575   -1.16   0.2487
## X16         -0.054446   0.125140   -0.44   0.6647
## X17          0.004335   0.112012    0.04   0.9692
## X18         -0.080796   0.109853   -0.74   0.4642
## X19         -0.085806   0.118553   -0.72   0.4713
## X20         -0.186006   0.104560   -1.78   0.0791 .
## X21          0.002111   0.108118    0.02   0.9845
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.999 on 79 degrees of freedom
## Multiple R-squared:  0.201,  Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.48
```

Multiple testing gives false positives

- 1 out of 20 variables significant at $\alpha = 0.05$
- 2 out of 20 variables significant at $\alpha = 0.1$
- Exactly the number of false positives we would expect.
- But notice the F-statistic: the variables are not **jointly** significant
- **Bonferroni correction**: use p-value cutoff α/m where m is the number of hypotheses.
    - Example: $0.05/20 = 0.0025$
    - Ensures that the family-wise error rate (probability of making at least 1 Type I error) is less than α.

4/ Linear Regression Model and Finite-sample Properties

Standard linear regression model

- Standard textbook model: **correctly specified linear CEF**
  - Designed for finite-sample results.

Assumption: Linear Regression Model

1. The variables $(Y, \mathbf{X})$ satisfy the the linear CEF assumption.

$$Y = \mathbf{X}'\boldsymbol{\beta} + e$$
$$\mathbb{E}[e \mid \mathbf{X}] = 0.$$

2. The design matrix is invertible $\mathbb{E}[\mathbf{X}\mathbf{X}'] > 0$ (positive definite).

- Basically this assumes the CEF of Y given $\mathbf{X}$ is linear.
- We continue to maintain $\{(Y_i, \mathbf{X}_i)\}$ are i.i.d.

Properties of OLS under linear CEF

- Linear CEFs imply stronger finite-sample guarantees:

1. **Unbiasedness:** $\mathbb{E}\left[\hat{\boldsymbol{\beta}} \mid \mathbb{X}\right] = \boldsymbol{\beta}$
2. **Conditional sampling variance**: let $\sigma_i^2 = \mathbb{E}[e_i^2 \mid \mathbf{X}_i]$

$$\mathbb{V}[\hat{\boldsymbol{\beta}} \mid \mathbb{X}] = (\mathbb{X}'\mathbb{X})^{-1} \left(\sum_{i=1}^{n} \sigma_i^2 \mathbf{X}_i \mathbf{X}_i' \right) (\mathbb{X}'\mathbb{X})^{-1}$$

- Useful when linearity holds by default (discrete X in experiments, etc)

Linear CEF under homoskedasticity

- Under homoskedasticity, we have a few other finite-sample results:

3. **Conditional sampling variance**: $\mathbb{V}[\hat{\boldsymbol{\beta}} \mid \mathbb{X}] = \sigma^2 \left(\mathbb{X}'\mathbb{X}\right)^{-1}$
4. **Unbiased variance estimator**: $\mathbb{E}\left[\widehat{\mathbb{V}}^0[\hat{\boldsymbol{\beta}}] \mid \mathbf{X}\right] = \sigma^2(\mathbb{X}'\mathbb{X})^{-1}$
5. **Gauss-Markov**: OLS is the best linear unbiased estimator of $\boldsymbol{\beta}$ (BLUE). If $\tilde{\boldsymbol{\beta}}$ is a linear estimator,

$$\mathbb{V}[\tilde{\boldsymbol{\beta}} \mid \mathbb{X}] \geq \mathbb{V}[\hat{\boldsymbol{\beta}} \mid \mathbb{X}] = \sigma^2 \left(\mathbb{X}'\mathbb{X}\right)^{-1}$$

  - For matrices, $\mathbf{A} \geq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semidefinite.
  - A matrix $\mathbf{C}$ is p.s.d. if $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$.
  - Upshot: OLS will have the smaller SEs than any other linear estimator.

Normal regression model

- Most parametric: $Y \sim \mathcal{N}(\mathbf{X}'\boldsymbol{\beta}, \sigma^2)$.
    - Normal error model since $e = Y - \mathbf{X}'\boldsymbol{\beta} \sim \mathcal{N}(0, \sigma^2)$.
- Rarely believed, but allows for exact inference for all n.
    - $(\hat{\beta}_j - \beta_j)/\widehat{\text{se}}(\hat{\beta}_j)$ follows a t distribution with $n - k$ degrees of freedom.
    - F statistics follows F distribution exactly rather than approximately.
- Software often implicitly assumes this for p-values.
- With reasonable n, asymptotic normality has the same effect.