15. Properties of Least Squares

Spring 2021

Matthew Blackwell

Gov 2002 (Harvard)

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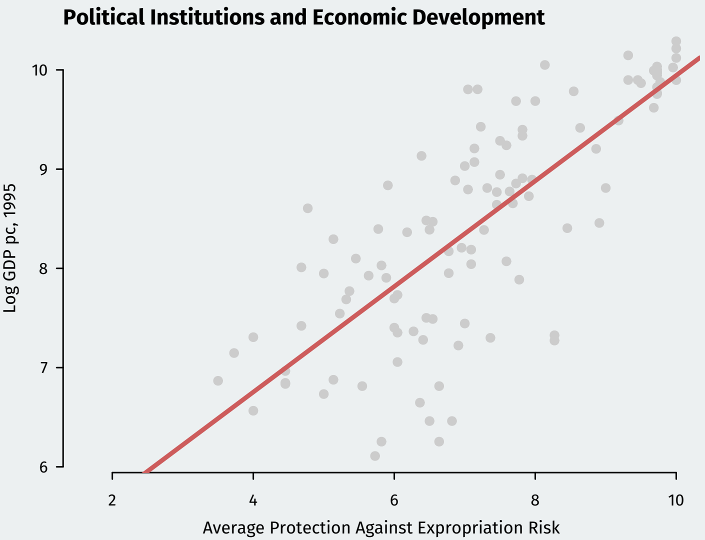
- Last time: OLS estimator, its algebraic properties.
- Now: its statistical properties, both finite-sample and asymptotic.

Acemoglu, Johnson, and Robinson (2001)



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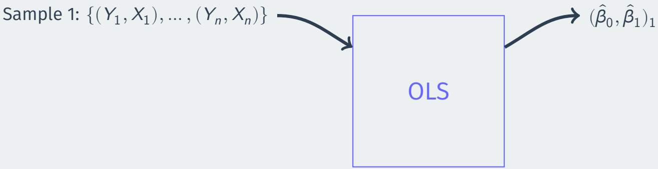

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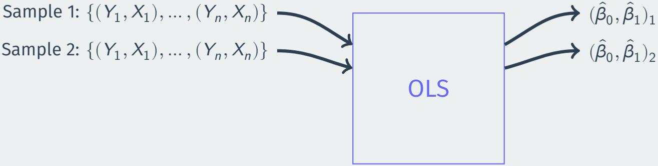

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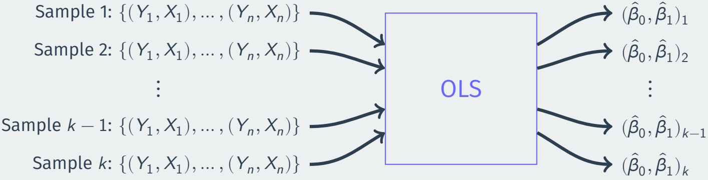

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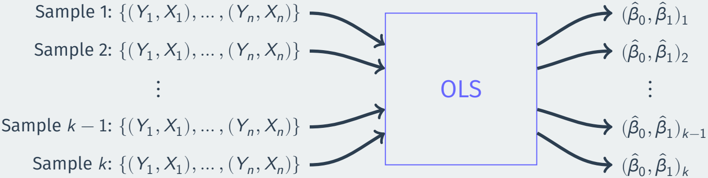

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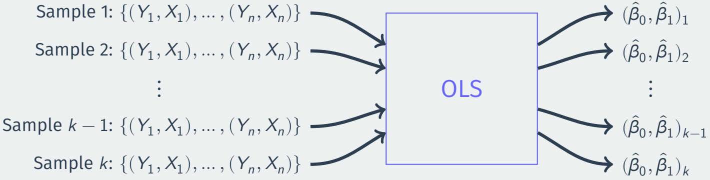

- Just like the sample mean or sample difference in means
- Has a sampling distribution, with a sampling variance/standard error.

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3. Plot the estimated regression line

Population Regression

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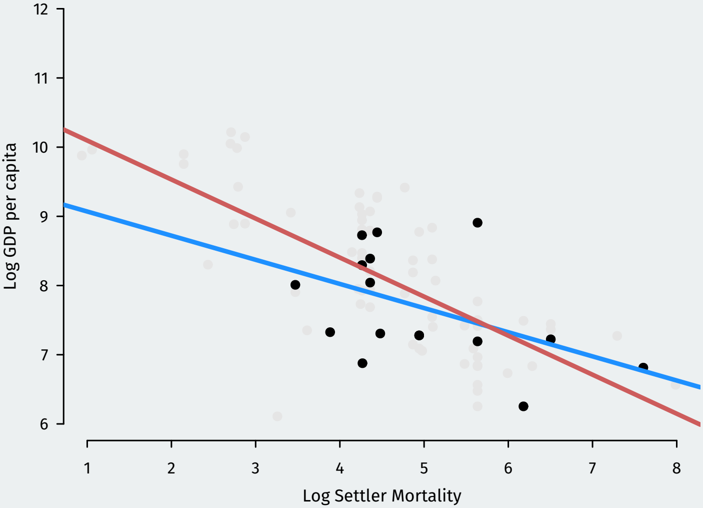

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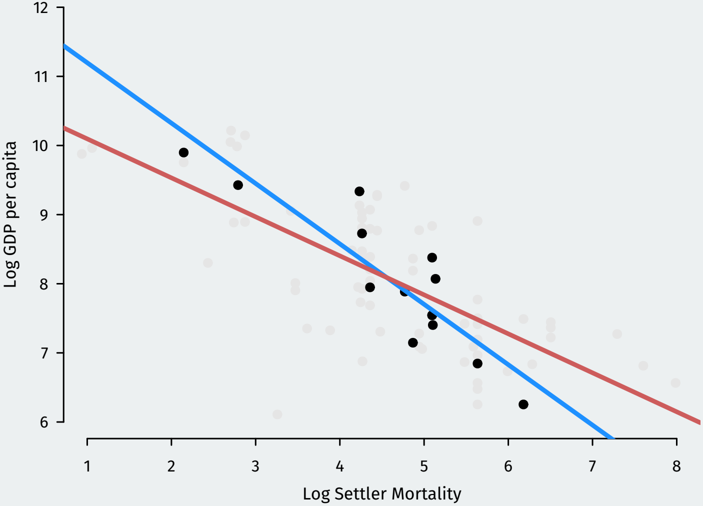

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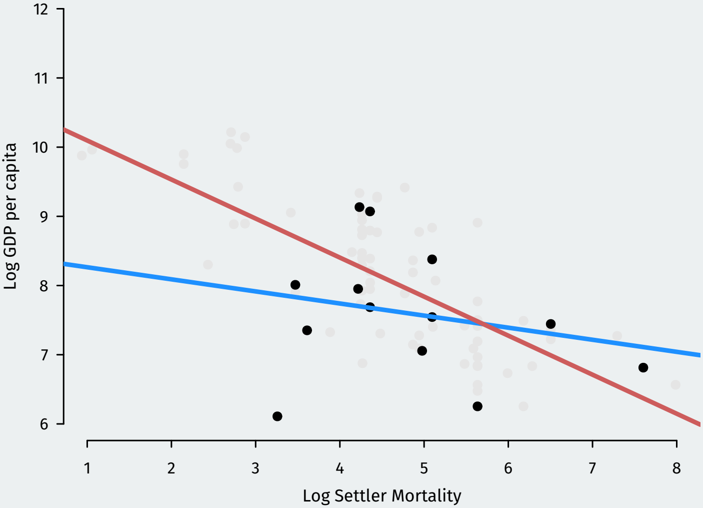

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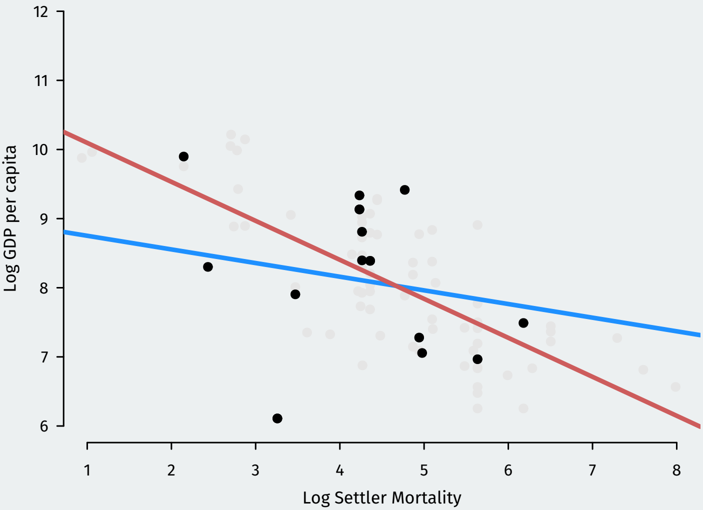

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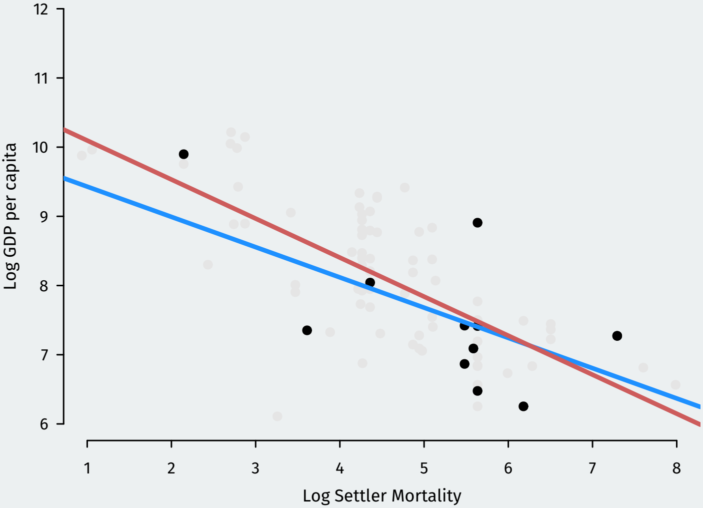

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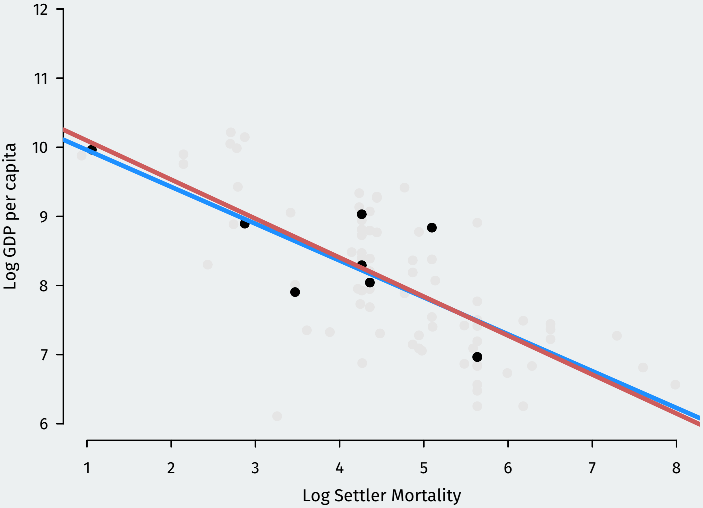

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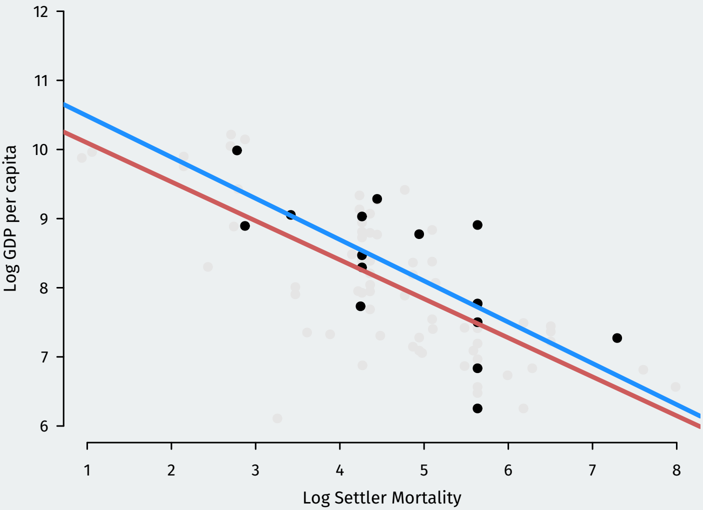

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    - **Linear regression/CEF model** for finite samples.

1/ Linear projection model and Large-sample Properties

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- What properties can we derive under such weak assumptions?

A very useful decomposition

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- $\mathbf{Q}_{\mathbf{XX}}$ is invertible by assumption, so by the continuous mapping theorem:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i'\right)^{-1} \xrightarrow{p} \mathbf{Q}_{\mathbf{XX}}^{-1} \quad \Longrightarrow \quad \hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta} + \mathbf{Q}_{\mathbf{XX}}^{-1}\cdot\mathbf{0} = \boldsymbol{\beta}$$

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- Valid with no restrictions on Y: could be binary, discrete, etc.
- Not guaranteed to be unbiased (unless CEF is linear, as we'll see...)

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- If $\mathbb{E}[g(\mathbf{X}_i)] = 0$, then we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} g(\mathbf{X}_i)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(\mathbf{X}_i) \overset{d}{\to} \mathcal{N}(0, \mathbb{E}[g(\mathbf{X}_i)g(\mathbf{X}_i)'])$$

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  - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.

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    - $n^{-1}\sum_{i=1}^{n}\mathbf{X}_i e_i$ is a sample average with $\mathbb{E}[\mathbf{X}_i e_i]=0$.
    - Rewrite as $\sqrt{n}$ times an average of i.i.d. mean-zero random vectors.
- Let $\boldsymbol{\Omega}=\mathbb{E}[e_i^2\mathbf{X}_i\mathbf{X}_i']$ and apply the CLT:

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i e_i\right)\overset{d}{\to}\mathcal{N}(0,\boldsymbol{\Omega})$$

Asymptotic normality

Theorem (Asymptotic Normality of OLS)

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- Allows us to formulate (approximate) confidence intervals, tests.

Estimating OLS variance

$$\mathbb{V}[\hat{\boldsymbol{\beta}}] = \frac{1}{n}\mathbf{V}_{\hat{\boldsymbol{\beta}}} = \mathbf{Q}_{\mathbf{XX}}^{-1}\boldsymbol{\Omega}\mathbf{Q}_{\mathbf{XX}}^{-1}$$

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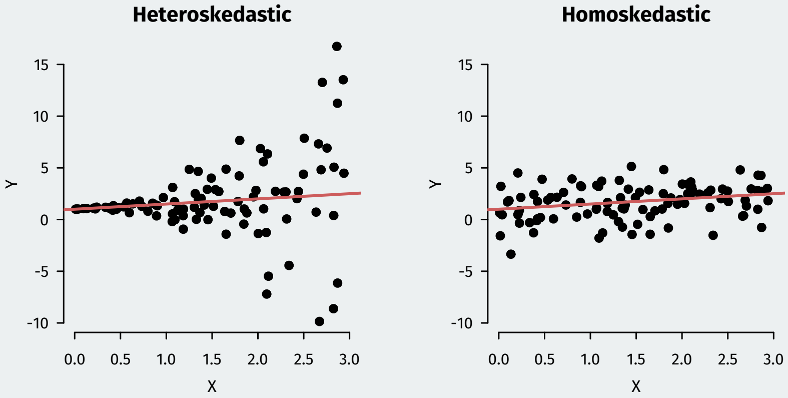
Homoskedasticity

Assumption: Homoskedasticity

The variance of the error terms is constant in $\mathbf{X}$, $\mathbb{E}[e^2 \mid \mathbf{X}] = \sigma^2(\mathbf{X}) = \sigma^2$.



Consequences of homoskedasticity

- Homoskedasticity implies $\mathbb{E}[e_i^2 \mathbf{X}_i \mathbf{X}_i'] = \mathbb{E}[e_i^2]\mathbb{E}[\mathbf{X}_i \mathbf{X}_i'] = \sigma^2 \mathbf{Q}_{\mathbf{XX}}$

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$$s^2 = \frac{1}{n-k}\sum_{i=1}^n \hat{e}_i^2 \qquad \widehat{\mathbb{V}}^0[\hat{\boldsymbol{\beta}}] = \frac{1}{n}s^2\left(\frac{1}{n}\sum_{i=1}^n \mathbf{X}_i\mathbf{X}_i'\right)^{-1} = s^2\left(\mathbb{X}'\mathbb{X}\right)^{-1}$$

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- LLN implies $s^2 \xrightarrow{p} \sigma^2$ and so $n\widehat{\mathbb{V}}^0[\hat{\boldsymbol{\beta}}]$ is consistent for $\mathbf{V}^0_{\hat{\boldsymbol{\beta}}}$

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- Software often uses t critical values instead of normal (we’ll see why).

2/ Inference for Multiple Parameters

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- Use the estimated covariance matrix:

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    - Compare model fit when we do and do not impose the null hypothesis.

Restricted and unrestricted models

- Estimates and SSR from **unrestricted model** (alternative is true):

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- If the null is true, we'd expect SSR_u to not be much bigger than SSR_r.

Restricted and unrestricted models

- Estimates and SSR from **unrestricted model** (alternative is true):

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- If the null is true, we'd expect SSR_u to not be much bigger than SSR_r.
    - SSR mechanically increases even if you add noise, but not that much.

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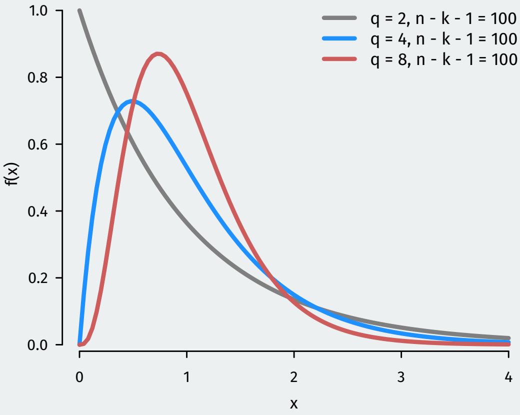
- Asymptotic distribution of F:

$$\frac{(SSR_r - SSR_u)/q}{SSR_u/(n-k-1)} \overset{d}{\to} F_{q,n-(k+1)}$$

F distribution



- Ratio of two χ^2 (Chi-squared) distributions

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- When applied to a single coefficient, equivalent to a t-test.

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- Often reported with regression output.

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- Illustration:
    - Randomly draw 21 variables independently.
    - Run a regression of the first variable on the rest.
- By design, no effect of any variable on any other.

Multiple test example

```
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.028039   0.113820   -0.25   0.8061
## X2          -0.150390   0.112181   -1.34   0.1839
## X3           0.079158   0.095028    0.83   0.4074
## X4          -0.071742   0.104579   -0.69   0.4947
## X5           0.172078   0.114002    1.51   0.1352
## X6           0.080852   0.108341    0.75   0.4577
## X7           0.102913   0.114156    0.90   0.3701
## X8          -0.321053   0.120673   -2.66   0.0094 **
## X9          -0.053122   0.107983   -0.49   0.6241
## X10          0.180105   0.126443    1.42   0.1583
## X11          0.166386   0.110947    1.50   0.1377
## X12          0.008011   0.103766    0.08   0.9387
## X13          0.000212   0.103785    0.00   0.9984
## X14         -0.065969   0.112214   -0.59   0.5583
## X15         -0.129654   0.111575   -1.16   0.2487
## X16         -0.054446   0.125140   -0.44   0.6647
## X17          0.004335   0.112012    0.04   0.9692
## X18         -0.080796   0.109853   -0.74   0.4642
## X19         -0.085806   0.118553   -0.72   0.4713
## X20         -0.186006   0.104560   -1.78   0.0791 .
## X21          0.002111   0.108118    0.02   0.9845
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.999 on 79 degrees of freedom
## Multiple R-squared:  0.201,  Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.48
```

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- 2 out of 20 variables significant at $\alpha = 0.1$
- Exactly the number of false positives we would expect.
- But notice the F-statistic: the variables are not **jointly** significant

3/ Linear Regression Model and Finite-sample Properties

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Assumption: Linear Regression Model

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$$Y = \mathbf{X}'\boldsymbol{\beta} + e$$
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- Basically this assumes the CEF of Y given $\mathbf{X}$ is linear.
- We continue to maintain $\{(Y_i, \mathbf{X}_i)\}$ are i.i.d.

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- Useful when linearity holds by default (discrete X in experiments, etc)

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  - A matrix $\mathbf{C}$ is p.s.d. if $\mathbf{x}'\mathbf{C}\mathbf{x} \geq 0$.
  - Upshot: OLS will have the smaller SEs than any other linear estimator.

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- With reasonable n, asymptotic normality has the same effect.